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A unified approach to decision-theoretic properties of the MLEs for the mean directions of several Langevin distributions

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ABSTRACT

The two-parameter Langevin distribution has been widely used for analyzing directional data. We address the problem of estimating the mean direction in its Cartesian and angular forms. The equivariant point estimation is introduced under different transformation groups. The maximum likelihood estimator (MLE) is shown to satisfy many decision theoretic properties such as admissibility, minimaxity, the best equivariance and risk-unbiasedness under various loss functions. Moreover, it is shown to be unique minimax when the concentration parameter is assumed to be known. These results extend and unify earlier results on the optimality of the MLE. These findings are also established for the problem of simultaneous estimation of mean directions of several independent Langevin populations. Further, estimation of a common mean direction of several independent Langevin populations is studied. A simulation study is carried out to analyze numerically the risk function of the MLE.

1. Introduction

Many real-life statistical problems can be modeled with the help of directional data in various scientific disciplines such as astronomy, geology and medicine. Due to lack of a preferred coordinate system on the unit hypersphere $\mathbb{S}_{p-1} = \{\boldsymbol{x} \in \mathbb{R}^p : \|\boldsymbol{x}\| = (\boldsymbol{x}^T\boldsymbol{x})^{1/2} = 1\}$ for representing the directional data, it is always desirable to choose an invariant model under the rotation for analyzing the directional data. The most popular choice of invariant distribution on $\mathbb{S}_{p-1}$ is a Langevin distribution (also called a von Mises–Fisher distribution), first introduced by Langevin [15] in statistical mechanics. For a detailed survey of inferential problems for directional distributions, one may refer to Jammalamadaka and SenGupta [12], Mardia and Jupp [19], Watson [24]. A random vector $\boldsymbol{X} \in \mathbb{S}_{p-1}$ has Langevin distribution $M_p(\boldsymbol{\mu}, \kappa)$ if its probability density with respect to the Lebesgue measure on $\mathbb{S}_{p-1}$ is given by

$$f_1(\boldsymbol{x}; \boldsymbol{\mu}, \kappa) = C_p(\kappa) \exp\{\kappa \boldsymbol{\mu}^T \boldsymbol{x}\}, \quad \boldsymbol{\mu} \in \mathbb{S}_{p-1},\ \kappa > 0, \tag{1.1}$$

with the mean direction $\boldsymbol{\mu}$ and the concentration parameter κ. The normalizing constant $C_p(\kappa)$ is equal to $(\kappa/2\pi)^{p/2} \left(\kappa I_{p/2-1}(\kappa)\right)^{-1}$, where I_v denotes the modified Bessel function of the first kind and of order v [2, p. 376, 9.6.19]. The Langevin distribution is referred to as circular normal or von Mises distribution for $p = 2$ and as Fisher distribution for $p = 3$.

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Sample mean direction is the most commonly used estimator for the mean direction of a rotationally symmetric distribution on $\mathbb{S}_{p-1}$. In fact, the Langevin distribution is the only rotationally symmetric distribution for which the maximum likelihood estimator (MLE) of the mean direction is the sample mean direction; see [24]. The asymptotic normality and consistency of the MLE for $\boldsymbol{\mu}$ were established in Watson [23]. Hendriks et al. [9] derived an asymptotic expression of order $O(n^{-3/2})$ for the expectation of the sample mean direction for a general density of form $f(\boldsymbol{\mu}^T\boldsymbol{x})$ when $E(\boldsymbol{X}) \neq \mathbf{0}$. For the Langevin model, Watamori [22] derived a similar expression with order $O(n^{-5/2})$.

For estimating $\boldsymbol{\mu}$, the following loss functions have been used in the literature:

$$L_1(\boldsymbol{\mu}, \boldsymbol{\delta}) = 1 - \boldsymbol{\mu}^T\boldsymbol{\delta}, \tag{1.2}$$

$$L_2(\boldsymbol{\mu}, \boldsymbol{\delta}) = 1 - \left(\boldsymbol{\mu}^T\boldsymbol{\delta}\right)^2, \tag{1.3}$$

though the loss L_2 has been studied only in [25]. The admissibility of the MLE for $\boldsymbol{\mu}$ was proved by Mardia and El-Atoum [18] under the loss L_1 for the case of known κ. In this case, Watson [25] established the admissibility and minimaxity of the Bayes estimator for the mean direction of a general density of form $f(\boldsymbol{\mu}^T\boldsymbol{x})$. He proved that the uniform distribution on $\mathbb{S}_{p-1}$ is the least favorable prior with respect to both loss functions L_1 and L_2. Hendriks [8] considered an extension of Langevin distribution on Stiefel manifolds. His results also imply the admissibility of the MLE of $\boldsymbol{\mu}$ under the loss L_1 with known κ.

A p-dimensional direction can also be specified by $p-1$ angular coordinates. We may represent the mean direction by its polar coordinates $\boldsymbol{\nu}^T = (\nu_1, \ldots, \nu_{p-1})$ (defined in the Appendix). We call $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as Cartesian and angular forms respectively of a Langevin mean direction. Sometimes in practice, it is more sensible to estimate the amplitude of a 2-dimensional direction than estimating a vector in $\mathbb{S}_1$. Similarly, for $p = 3$, the geographical coordinates, longitude and latitude (or colatitude), may be more useful than a unit vector in $\mathbb{R}^3$. It may be noted, however, that the longitude is not unique on the north pole as well as on the south pole. This problem occurs specially when we go for angular representation of a direction in the higher dimensions $p \geq 3$. Further details are given in the Appendix.

We consider estimation of $\boldsymbol{\nu}$ under the following loss functions which are the angular versions of L_1 and L_2 in polar coordinates $\boldsymbol{\xi}^T = (\xi_1, \ldots, \xi_{p-1})$ of $\boldsymbol{\delta}$:

$$L_3(\boldsymbol{\nu}, \boldsymbol{\xi}) = 1 - \boldsymbol{u}^T(\boldsymbol{\nu})\,\boldsymbol{u}(\boldsymbol{\xi}), \tag{1.4}$$

$$L_4(\boldsymbol{\nu}, \boldsymbol{\xi}) = 1 - \left(\boldsymbol{u}^T(\boldsymbol{\nu})\,\boldsymbol{u}(\boldsymbol{\xi})\right)^2, \tag{1.5}$$

where $\boldsymbol{u}(.)$ is as defined in the Appendix.

The estimation of $\boldsymbol{\nu}$ is studied by many authors for the case $p = 2$; see, e.g., [1,7]. For a general p, SenGupta and Maitra [21] established the admissibility of MLE under the loss function L_3. They also proved its minimaxity and the best rotation equivariance for $p = 2$ applying the results of [13]. The question of existence of the best rotation equivariant estimator for $p \geq 3$ was left open. In this paper, we obtain the best rotation equivariant estimator for a general p. This is also possible by solving the problem for the Cartesian representation of the mean direction and noting that the problems of estimating $\boldsymbol{\mu}$ under the loss functions L_1 and L_2 are equivalent to those of estimating $\boldsymbol{\nu}$ under the loss functions L_3 and L_4 respectively from a decision theoretic point of view.

In multi-sample problem, occasionally several independent Langevin models can share their mean directions. Holmquist [10,11] considered this estimation problem for the cases of $p = 2$ and general p respectively when concentration parameters of populations are assumed to be known. He derived the MLE and a test procedure for the common mean direction. Fisher and Lewis [6] introduced pooled estimators for the common mean direction of general circular and spherical distributions.

Let $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$ denote the mean directions of m (≥ 2) independent Langevin distributions. Define $\tilde{\boldsymbol{\mu}}^T = (\boldsymbol{\mu}_1^T, \ldots, \boldsymbol{\mu}_m^T)$ and $\tilde{\boldsymbol{\delta}}^T = (\boldsymbol{\delta}_1^T, \ldots, \boldsymbol{\delta}_m^T)$. For estimating $\tilde{\boldsymbol{\mu}}$ by $\tilde{\boldsymbol{\delta}}$, we consider loss functions L_5 and L_6 as defined below. These are extensions of L_1 and L_2 respectively.

$$L_5(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\delta}}) = m - \tilde{\boldsymbol{\mu}}^T\tilde{\boldsymbol{\delta}} \tag{1.6}$$

$$L_6(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\delta}}) = m - \sum_{i=1}^{m}\left(\boldsymbol{\mu}_i^T\boldsymbol{\delta}_i\right)^2. \tag{1.7}$$

This simultaneous estimation problem (in angular version) was considered by SenGupta and Maitra [21] for the case when the populations are assumed to have a common concentration parameter. The sample sizes were also taken to be equal. They established the admissibility of the MLE under the angular version of loss L_5. They also proved the best rotation equivariance of the MLE only for the case of $p = 2$.

In this article, we unify results on decision theoretic properties of the MLE(s) for Langevin mean direction(s). In Section 2, we introduce the notation for the MLE of a Langevin mean direction in its Cartesian form. First we consider the one sample problem. The equivariant estimation is studied with respect to two transformation groups. Sections 3 and 4 are devoted to the problem of estimating the mean direction in its Cartesian form under loss functions L_1 and L_2. The cases of known and unknown κ are probed separately. In Section 5, we investigate the problem of estimating the common mean direction of several Langevin populations. In Section 6, the problem of simultaneous estimation of several Langevin mean directions is