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Multivariate and multiradial Schoenberg measures with their dimension walks

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ABSTRACT

The paper fixes some important properties of matrix-valued correlation functions associated to Multivariate Gaussian fields in a Euclidean space $\mathbb{R}^d$. In particular, we focus (a) on the isotropic (radially symmetric) case and (b) on anisotropy obtained through isotropy between components of the lag vector. This second case includes, as special case, space–time and fully symmetric correlation functions.

Starting from the multivariate analogue of the Schoenberg integral representation of isotropic correlation functions, we characterize their associated measures, which are called here m-Schoenberg measures. We also propose a new dimension walk for the componentwise isotropic case. Finally, we obtain examples where dimension walks for multivariate correlations are not well defined.

1. Introduction

Correlation functions are crucial to geostatistics and the Gaussian framework gives them a central role for spatial and space–time modelling. The recent literature (Gneiting et al. [17]; Apanasovich et al. [2]; Porcu et al. [22] and Daley et al. [7] with the references therein) shows the importance of matrix-valued correlations associated to multivariate Gaussian fields. In particular, there is an increasing number of applications to environmental and climatological data, image analysis and many other fields, where multivariate correlations are needed. The recent survey by Genton and Kleiber [11] gives further evidences of the importance of multivariate correlations for many branches of applied sciences.

The assumption of isotropy or radial symmetry has become ubiquitous in spatial statistics, although it can be restrictive for modelling some natural phenomena. Schoenberg representation of isotropic positive definite functions characterizes them as being the scale mixtures of the characteristic function of a random vector being uniformly distributed on the spherical shell of $\mathbb{R}^d$ with measures defined on the positive real line [28].

This in turn allows to focus on the following facts: (a) on the basis of Schoenberg's representation, Matheron [21], Wendland [30] and Gneiting [13,16] found bijections (projection operators) that allow to map a positive definite function in $\mathbb{R}^d$ onto positive definite functions in $\mathbb{R}^{d'}$, with $d \neq d'$. These projection operators are then crucial in order to (a.1) simulate Gaussian fields (through the so called turning band operator as introduced in Matheron [21]) and (a.2) improve the differentiability at the origin of a given positive definite function (the so called Montée operator), this last aspect being of particular importance for tapering techniques, as summarized in Gneiting [16].

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(b) Daley and Porcu [6] focus on the spectral measure associated to functions admitting the Schoenberg representation and adopt the illustrative name *Schoenberg measures*. In particular, they show the relations between these measures and the projection operators above. A relevant fact is that these measures are crucial for inspecting the equivalence of Gaussian measures indexed by given isotropic covariances, which in turn is the crux for the arguments of many results proposed under infill asymptotics [29], as well as for many results on asymptotic effects of tapering on estimation and prediction, and we refer the reader to the survey in [11] with the references therein.

Given the interest of the recent literature on multivariate covariance functions, it seems natural to reconsider the questions arising from points (a) and (b), this time in the multivariate case: equivalence of Gaussian measures for vector fields has been addressed in Ruiz-Medina and Porcu –[26] –; multivariate tapers have been proposed in [25,7]. In order to assess the properties of multivariate tapering techniques, we need further extensions to the multivariate case of the results in points (a) and (b), with special emphasis to (i) projection operators associated to multivariate and radially symmetric covariances, (ii) simulation methods for multivariate Gaussian fields and (iii) characterization of the (matrix-valued) Schoenberg measures associated to multivariate covariances. In this paper we give special attention to this last aspect, which requires a considerable mathematical effort, and which opens a clear bridge with the results on equivalence of Gaussian measures offered in [26].

Another aspect considered in this paper is that of multiradial positive definite functions, which allow to model anisotropic fields (through isotropy between components) as well space–time covariances being spatially isotropic and temporally symmetric, for which the associated Schoenberg-type representation has been shown in Porcu et al. [23], as well as some projection operators being the multiradial analogue of those listed in point (a) above. In this paper, we extend these results to the matrix-valued case, both in terms of integral representation, projection operators and multivariate Schoenberg measures.

Finally, we also find new projection operators for multiradial functions, being walk through dimensions based on Beta functions.

The remainder of the paper is organized as follows: Section 2 contains expository material and necessary background. Section 3 is dedicated to the case of isotropic matrix-valued covariance functions. Section 4 generalizes the result in Section 3 to the componentwise isotropic case. In particular, we introduce the multiradial version of the Descente and Montée etendue as in Porcu et al. [23]. In Section 5 we present two operators supporting dimension walks for multiradial matrix-valued correlation functions. Appendix contains some examples, some technical lemmas with their proofs and the proofs of Lemma 1 and Proposition 6, this is done for a neater exposition of the results.

2. Background and basic definitions

This section contains expository material which will allow for a self contained, and neater exposition, of what follows.

We consider the set $\exp(\mathbb{Z}_+) = \mathbb{Z}_+ \cup \mathbb{Z}_+^2 \cup \mathbb{Z}_+^3 \ldots$ (disjoint union). An element $\mathbf{d} \in \exp(\mathbb{Z}_+)$ is expressed as $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n$ for $n \geq 1$, the dimension and the length of $\mathbf{d} \in \exp(\mathbb{Z}_+)$ are defined as $n(\mathbf{d}) = n$ and $|\mathbf{d}| = \sum_{i=1}^n d_i$, respectively. For $\mathbf{d},\ \mathbf{d}^* \in \exp(\mathbb{Z}_+)$, we write $\mathbf{d} \leq \mathbf{d}^*$ if and only if $n(\mathbf{d}) = n(\mathbf{d}^*)$ and $d_i \leq d_i^*$ for all $i = 1, 2, \ldots, n(\mathbf{d})$. The vectors with their components equal are denoted in bold math $\mathbf{0}$, $\mathbf{1}$, . . ., when the dimension is unambiguous.

Let M_m denote the set of $m \times m$-dimensional complex-valued matrices. A mapping $\mathbf{C} = [C_{ij}] : \mathbb{R}^d \times \mathbb{R}^d \to M_m$ is positive definite if for any finite dimensional collection of points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of $\mathbb{R}^d$ and the same number of m-dimensional vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{C}^m$, $\mathbf{c}_i = (c_{i1}, \ldots, c_{im})'$, the following inequality holds:

$$\sum_{j,k}^{n} \sum_{i,l}^{m} c_{ji}\bar{c}_{kl} C_{il}(\mathbf{x}_j, \mathbf{x}_k) \geq 0. \tag{1}$$

Kolmogorov's existence theorem implies that, for any positive definite mapping $\mathbf{C}$ as just defined, there exists a Gaussian vector-valued random field – for short RF –, $\mathbf{Z}(\mathbf{x}) = (Z_1(\mathbf{x}), \ldots, Z_m(\mathbf{x}))'$ on $\mathbb{R}^d$ such that

$$\mathrm{Cov}\,(\mathbf{Z}(\mathbf{x}), \mathbf{Z}(\mathbf{y})) = \mathbf{C}(\mathbf{x}, \mathbf{y}) = \left[C_{ij}(\mathbf{x}, \mathbf{y})\right]_{i,j=1}^m, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Under the assumption of weak stationarity, we have $\mathbf{C}(\mathbf{x}, \mathbf{y}) = \mathbf{K}(\mathbf{y} - \mathbf{x})$ for some mapping $\mathbf{K} : \mathbb{R}^d \to M_m$. We call $\Phi_{\mathbf{d}}^m$ the class of matrix-valued $\mathbf{d}$-anisotropic correlation functions, i.e. $\Phi_{\mathbf{d}}^m$ is the class of matrix-valued functions $\boldsymbol{\varphi} = [\varphi_{ij}]_{i,j=1}^m$ for which $\varphi_{ij} : [0, \infty)^n \to \mathbb{R}$ is continuous, $\varphi_{ii}(\mathbf{0}) = 1$ $(i = 1, \ldots, m)$, and such that there exists a stationary Gaussian m-variate RF $\{\mathbf{Z}(\mathbf{x}) = (Z_1(\mathbf{x}), Z_2(\mathbf{x}), \ldots, Z_m(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^d, \mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}\}$ with matrix-valued covariance

$$\begin{aligned}\mathrm{Cov}\,(\mathbf{Z}(\mathbf{x}), \mathbf{Z}(\mathbf{x} + \boldsymbol{\tau})) &= \mathbf{K}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n) = \left[K_{ij}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)\right]_{i,j=1}^m \\ &= \mathrm{diag}\{\boldsymbol{\sigma}\}\, \boldsymbol{\varphi}(\|\boldsymbol{\tau}_1\|, \ldots, \|\boldsymbol{\tau}_n\|)\, \mathrm{diag}\{\boldsymbol{\sigma}\} \\ &= \mathrm{diag}\{\boldsymbol{\sigma}\}\, \left[\varphi_{ij}(\|\boldsymbol{\tau}_1\|, \ldots, \|\boldsymbol{\tau}_n\|)\right]_{i,j=1}^m\, \mathrm{diag}\{\boldsymbol{\sigma}\},\end{aligned} \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm, $\boldsymbol{\sigma}$ is an m-vector of the nonnegative elements σ_j for which $\mathrm{Var}\, Z_j(\mathbf{x}) = \sigma_j^2$ (all $\mathbf{x} \in \mathbb{R}^d$) and $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$. Our notation is consistent with the notation given in previous works, e.g. Daley