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Nonparametric functional central limit theorem for time series regression with application to self-normalized confidence interval

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ABSTRACT

This paper is concerned with the inference of nonparametric mean function in a time series context. The commonly used kernel smoothing estimate is asymptotically normal and the traditional inference procedure then consistently estimates the asymptotic variance function and relies upon normal approximation. Consistent estimation of the asymptotic variance function involves another level of nonparametric smoothing. In practice, the choice of the extra bandwidth parameter can be difficult, the inference results can be sensitive to bandwidth selection and the normal approximation can be quite unsatisfactory in small samples leading to poor coverage. To alleviate the problem, we propose to extend the recently developed self-normalized approach, which is a bandwidth free inference procedure developed for parametric inference, to construct point-wise confidence interval for nonparametric mean function. To justify asymptotic validity of the self-normalized approach, we establish a functional central limit theorem for recursive nonparametric mean regression function estimates under primitive conditions and show that the limiting process is a Gaussian process with non-stationary and dependent increments. The superior finite sample performance of the new approach is demonstrated through simulation studies.

1. Introduction

Nonparametric methods are useful complements to the traditional well developed parametric counterparts. They allow the users to entertain model flexibility while reducing modeling bias, and partly due to this reason, nonparametric inference has been extensively studied. This paper concerns a new way of addressing nonparametric inference in the time series setting. There is a huge literature about the use of nonparametric methods in time series analysis, and asymptotic theory for nonparametric estimators and tests has been quite well developed for weakly dependent time series data. We refer the reader to Chapters 5–10 in [3] for a nice introduction of some basic ideas and results.

Given stationary time series $\{(X_i, Y_i)\}_{i=1}^n$, we focus on inference for the conditional mean function $\mu(x) = \mathbb{E}(Y_i|X_i = x)$; see Section 4 for some possible extensions to other nonparametric functions. Let $\hat{\mu}_n(x)$ be a nonparametric estimate of $\mu(x)$

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based on the full sample. Under suitable regularity and weak dependence conditions, we have

$$\sqrt{nb_n}\frac{\hat{\mu}_n(x) - \mu(x) - b_n^2 r(x)}{s(x)} \xrightarrow{d} N(0, 1), \tag{1}$$

where b_n is an appropriate bandwidth, $b_n^2 r(x)$ is the bias term, $s^2(x)$ is the asymptotic variance function, and $\xrightarrow{d}$ stands for convergence in distribution. To construct a point-wise confidence interval for $\mu(x)$, the traditional approach involves consistent estimation of $s^2(x)$ through an extra nonparametric smoothing procedure which inevitably introduces estimation error. The latter issue becomes even more serious when $s(x) \approx 0$ so that the left hand side of (1) is very sensitive to the estimation error of $s(x)$. In particular, even if the absolute estimation error is small, the relative estimation error can be large, which leads to poor coverage in the constructed confidence interval. Thus, one needs to deal with the unpleasant phenomenon that, the smaller $s(x)$ (i.e. lower noise level), the more difficult to carry out statistical inference. Furthermore, nonparametric estimation of $s(x)$ involves extra bandwidth parameter(s). Two users using two different bandwidths in estimating $s(x)$ for the same data set may get quite different results.

To alleviate the above-mentioned problem in the traditional inference procedure, we propose to extend the recently developed self-normalized (SN, hereafter) approach [14] to nonparametric setting. The SN approach was developed for a finite dimensional parameter of a stationary time series and it has the nice feature of being bandwidth free. The basic idea of the SN approach, when applied to nonparametric setting, is to use estimates of $\mu(x)$ on the basis of recursive subsamples to form a self-normalizer that is an inconsistent estimator of $s(x)$. Although it is inconsistent, the self-normalizer is proportional to $s(x)$, and the limiting distribution of the self-normalized quantity is pivotal. The literature on the SN approach and related methods [10,12,8,7,14,15,17,21] has been growing recently, but most of the work is limited to parametric inference, where the parameter of interest is finite dimensional and the method of estimation does not involve smoothing. Kim and Zhao [9] studied SN approach for the nonparametric mean function in longitudinal models, but the data are essentially independent due to the independent subjects. To the best of our knowledge, the SN-based extension to nonparametric time series inference seems new.

An important theoretical contribution of this article is that we establish nonparametric functional central limit theorem (FCLT, hereafter) of some recursive estimates of $\mu(\cdot)$ under primitive conditions. To be specific, denote by $\hat{\mu}_m(x)$ the nonparametric estimate of $\mu(x)$ using data $\{(X_i, Y_i)\}_{i=1}^m$ up to time m and bandwidth b_m. Throughout, denote by $\lfloor v \rfloor$ the integer part of v. We show that, due to the sample-size-dependent bandwidths, the process $\{\hat{\mu}_{\lfloor nt \rfloor}(x) - \mu(x)\}$ indexed by t, after proper normalization, converges weakly to a Gaussian process $\{G_t\}$ with non-stationary and dependent increments. Such a result is very different from the FCLT required for the SN approach in the parametric inference problems, where the limiting process is a Brownian motion with stationary and independent increments.

Throughout, we write $\xi \in \mathcal{L}^p$ $(p \geq 1)$ if $\|\xi\|_p := (\mathbb{E}|\xi|^p)^{1/p} < \infty$. The symbols $O_p(1)$ and $o_p(1)$ signify being bounded in probability and convergence to zero in probability, respectively. For sequences $\{a_n\}$ and $\{c_n\}$, write $a_n \asymp c_n$ if $a_n/c_n \to 1$. The article is organized as follows. Section 2 presents the main results, including the FCLT for nonparametric recursive estimates and the self-normalization based confidence interval. Simulation results are presented in Section 3. Section 4 concludes and technical details are gathered in the Appendix.

2. Main results

We consider the nonparametric mean regression model:

$$Y_i = \mu(X_i) + e_i, \tag{2}$$

where $\mu(\cdot)$ is the nonparametric mean function of interest and $\{e_i\}$ are noises. As an important special case, let $X_i = Y_{i-1}$ and $e_i = \sigma(X_i)\varepsilon_i$ for innovations $\{\varepsilon_i\}$ and a scale function $\sigma(\cdot)$, then we have the nonparametric autoregressive (AR) model $Y_i = \mu(Y_{i-1}) + \sigma(Y_{i-1})\varepsilon_i$, which includes many nonlinear time series models, such as linear AR, threshold AR, exponential AR, and AR with conditional heteroscedasticity; see [3]. We assume that $\{(X_i, Y_i)\}_{i=1}^n$ are stationary time series observations so that they have a natural ordering in time, i.e., (X_i, Y_i) is the observation at time i.

2.1. Nonparametric FCLT for recursive estimates

Throughout let x be a fixed interior point in the support of X_i. Denote by $\hat{\mu}_m(x)$ the nonparametric estimate of $\mu(x)$ based on data $\{(X_i, Y_i)\}_{i=1}^m$ up to time m. In this paper we consider the local linear kernel smoothing estimator [2] of $\mu(x)$:

$$\hat{\mu}_m(x) = \hat{a}_0, \qquad (\hat{a}_0, \hat{a}_1) = \underset{(a_0,a_1)}{\operatorname{argmin}} \sum_{i=1}^m \Big[Y_i - a_0 - a_1(X_i - x)\Big]^2 K\Big(\frac{X_i - x}{b_m}\Big), \tag{3}$$

where $K(\cdot)$ is a kernel function and $b_m > 0$ is the bandwidth. By elementary calculation,

$$\hat{\mu}_m(x) = \frac{M_m(2)N_m(0) - M_m(1)N_m(1)}{M_m(2)M_m(0) - M_m(1)^2}, \tag{4}$$