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# Statistical inference for 2-type doubly symmetric critical irreducible continuous state and continuous time branching processes with immigration

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ABSTRACT

We study asymptotic behavior of conditional least squares estimators for 2-type doubly symmetric critical irreducible continuous state and continuous time branching processes with immigration based on discrete time (low frequency) observations.



## 1. Introduction

Asymptotic behavior of conditional least squares (CLS) estimators for critical continuous state and continuous time branching processes with immigration (CBI processes) is available only for single-type processes. Huang et al. [12] considered a single-type CBI process which can be represented as a pathwise unique strong solution of the stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t (\beta + \widetilde{b} X_s)\, \mathrm{d}s + \int_0^t \sqrt{2c \max\{0, X_s\}}\, \mathrm{d}W_s + \int_0^t \int_0^\infty \int_0^\infty z \mathbb{1}_{\{u \leqslant X_{s-}\}} \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_0^\infty z\, M(\mathrm{d}s, \mathrm{d}z) \tag{1.1}$$

for $t \in [0, \infty)$, where $\beta, c \in [0, \infty)$, $\widetilde{b} \in \mathbb{R}$, and $(W_t)_{t \geqslant 0}$ is a standard Wiener process, $N$ and $M$ are independent Poisson random measures on $(0, \infty)^3$ and on $(0, \infty)^2$ with intensity measures $\mathrm{d}s\, \mu(\mathrm{d}z)\, \mathrm{d}u$ and $\mathrm{d}s\, \nu(\mathrm{d}z)$, respectively, $\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) := N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}s\, \mu(\mathrm{d}z)\, \mathrm{d}u$ is the compensated Poisson random measure corresponding to $N$, the measures $\mu$ and $\nu$ satisfy some moment conditions, and $(W_t)_{t \geqslant 0}$, $N$ and $M$ are independent. The model is called subcritical, critical or supercritical if $\widetilde{b} < 0$, $\widetilde{b} = 0$ or $\widetilde{b} > 0$, see Huang et al. [12, p. 1105] or Definition 2.8. Based on discrete time (low frequency)

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observations $(X_k)_{k\in\{0,1,\ldots,n\}}$, $n \in \{1, 2, \ldots\}$, Huang et al. [12] derived weighted CLS estimator of $(\widetilde{b}, \beta)$. Under some second order moment assumptions, supposing that $c$, $\mu$ and $\nu$ are known, they showed the following results: in the subcritical case the estimator of $(\widetilde{b}, \beta)$ is asymptotically normal; in the critical case the estimator of $\widetilde{b}$ has a non-normal limit, but the asymptotic behavior of the estimator of $\beta$ remained open; in the supercritical case the estimator of $\widetilde{b}$ is asymptotically normal with a random scaling, but the estimator of $\beta$ is not weakly consistent.

Based on the observations $(X_k)_{k\in\{0,1,\ldots,n\}}$, $n \in \{1, 2, \ldots\}$, supposing that $c$, $\mu$ and $\nu$ are known, Barczy et al. [4] derived (non-weighted) CLS estimator $(\widehat{\widetilde{b}}_n, \widehat{\widetilde{\beta}}_n)$, of $(\widetilde{b}, \widetilde{\beta})$, where $\widetilde{\beta} := \beta + \int_0^\infty z\,\nu(\mathrm{d}z)$. In the critical case, under some moment assumptions, it has been shown that $\big(n(\widehat{\widetilde{b}}_n - \widetilde{b}), \widehat{\widetilde{\beta}}_n - \widetilde{\beta}\big)$ has a non-normal limit. As a by-product, the estimator $\widehat{\widetilde{\beta}}_n$ is not weakly consistent.

Overbeck and Rydén [21] considered CLS and weighted CLS estimators for the well-known Cox–Ingersoll–Ross model, which is, in fact, a single-type diffusion CBI process (without jump part), i.e., when $\mu = 0$ and $\nu = 0$ in (1.1). Based on discrete time observations $(X_k)_{k\in\{0,1,\ldots,n\}}$, $n \in \{1, 2, \ldots\}$, they derived CLS estimator of $(\widetilde{b}, \beta, c)$ and proved its asymptotic normality in the subcritical case. Note that Li and Ma [20] started to investigate the asymptotic behavior of the CLS and weighted CLS estimators of the parameters $(\widetilde{b}, \beta)$ in the subcritical case for a Cox–Ingersoll–Ross model driven by a stable noise, which is again a special single-type CBI process (with jump part).

In this paper we consider a 2-type CBI process which can be represented as a pathwise unique strong solution of the SDE

$$\begin{aligned}\boldsymbol{X}_t &= \boldsymbol{X}_0 + \int_0^t (\boldsymbol{\beta} + \widetilde{\boldsymbol{B}}\boldsymbol{X}_s)\,\mathrm{d}s + \sum_{i=1}^2 \int_0^t \sqrt{2c_i \max\{0, X_{s,i}\}}\,\mathrm{d}W_{s,i}\,\boldsymbol{e}_i \\ &\quad + \sum_{j=1}^2 \int_0^t \int_{\mathcal{U}_2} \int_0^\infty \boldsymbol{z}\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,\widetilde{N}_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) + \int_0^t \int_{\mathcal{U}_2} \boldsymbol{z}\,M(\mathrm{d}s, \mathrm{d}\boldsymbol{z}) \end{aligned} \tag{1.2}$$

for $t \in [0, \infty)$. Here $X_{t,i}$, $i \in \{1, 2\}$, denotes the coordinates of $\boldsymbol{X}_t$, $\boldsymbol{\beta} \in [0, \infty)^2$, $\widetilde{\boldsymbol{B}} \in \mathbb{R}^{2\times 2}$ has non-negative off-diagonal entries, $c_1, c_2 \in [0, \infty)$, $\boldsymbol{e}_1, \boldsymbol{e}_2$ denotes the natural basis in $\mathbb{R}^2$, $\mathcal{U}_2 := [0, \infty)^2 \setminus \{(0, 0)\}$, $(W_{t,1})_{t\geqslant 0}$ and $(W_{t,2})_{t\geqslant 0}$ are independent standard Wiener processes, $N_j, j \in \{1, 2\}$, and $M$ are independent Poisson random measures on $(0, \infty) \times \mathcal{U}_2 \times (0, \infty)$ and on $(0, \infty) \times \mathcal{U}_2$ with intensity measures $\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u, j \in \{1, 2\}$, and $\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{z})$, respectively, $\widetilde{N}_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) := N_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) - \mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u, j \in \{1, 2\}$. We suppose that the Borel measures $\mu_j, j \in \{1, 2\}$, and $\nu$ on $\mathcal{U}_2$ satisfy some moment conditions, and $(W_{t,1})_{t\geqslant 0}, (W_{t,2})_{t\geqslant 0}, N_1, N_2$ and $M$ are independent. We will suppose that the process $(\boldsymbol{X}_t)_{t\geqslant 0}$ is doubly symmetric in the sense that

$$\widetilde{\boldsymbol{B}} = \begin{bmatrix} \gamma & \kappa \\ \kappa & \gamma \end{bmatrix},$$

where $\gamma \in \mathbb{R}$ and $\kappa \in [0, \infty)$. Note that the parameters $\gamma$ and $\kappa$ might be interpreted as the transformation rates of one type to the same type and one type to the other type, respectively, compare with Xu [23]; that is why the model can be called doubly symmetric.

The model will be called subcritical, critical or supercritical if $s < 0$, $s = 0$ or $s > 0$, respectively, where $s := \gamma + \kappa$ denotes the criticality parameter, see Definition 2.8.

For the simplicity, we suppose $\boldsymbol{X}_0 = (0, 0)^\top$. We suppose that $c_1, c_2, \mu_1, \mu_2$ and $\nu$ are known, and we derive the CLS estimators of the parameters $s, \gamma, \kappa$ and $\boldsymbol{\beta}$ based on discrete time observations $(\boldsymbol{X}_k)_{k\in\{1,\ldots,n\}}$, $n \in \{1, 2, \ldots\}$. In the irreducible and critical case, i.e., when $\kappa > 0$ and $s = \gamma + \kappa = 0$, under some moment conditions, we describe the asymptotic behavior of these CLS estimators as $n \to \infty$, provided that $\boldsymbol{\beta} \neq (0, 0)^\top$ or $\nu \neq 0$, see Theorem 3.1. We point out that the limit distributions are non-normal in general. In the present paper we do not investigate the asymptotic behavior of CLS estimators of $s, \gamma, \kappa$ and $\boldsymbol{\beta}$ in the subcritical and supercritical cases, it could be the topic of separate papers, needing different approaches.

Xu [23] considered a 2-type diffusion CBI process (without jump part), i.e., when $\mu_j = 0, j \in \{1, 2\}$, and $\nu = 0$ in (1.2). Based on discrete time (low frequency) observations $(\boldsymbol{X}_k)_{k\in\{1,\ldots,n\}}$, $n \in \{1, 2, \ldots\}$, Xu [23] derived CLS estimators and weighted CLS estimators of $(\boldsymbol{\beta}, \widetilde{\boldsymbol{B}}, c_1, c_2)$. Provided that $\boldsymbol{\beta} \in (0, \infty)^2$, the diagonal entries of $\widetilde{\boldsymbol{B}}$ are negative, the off-diagonal entries of $\widetilde{\boldsymbol{B}}$ are positive, the determinant of $\widetilde{\boldsymbol{B}}$ is positive and $c_i > 0, i \in \{1, 2\}$ (which yields that the process $\boldsymbol{X}$ is irreducible and subcritical, see Xu [23, Theorem 2.2] and Definitions 2.7 and 2.8), it was shown that these CLS estimators are asymptotically normal, see Theorem 4.6 in Xu [23].

Finally, we give an overview of the paper. In Section 2, for completeness and better readability, from Barczy et al. [8,9], we recall some notions and statements for multi-type CBI processes such as the form of their infinitesimal generator, Laplace transform, a formula for their first moment, the definition of subcritical, critical and supercritical irreducible CBI processes, see Definitions 2.7 and 2.8. We recall a result due to Barczy and Pap [9, Theorem 4.1] stating that, under some fourth order moment assumptions, a sequence of scaled random step functions $(n^{-1}\boldsymbol{X}_{\lfloor nt\rfloor})_{t\geqslant 0}$, $n \geqslant 1$, formed from a critical, irreducible multi-type CBI process $\boldsymbol{X}$ converges weakly towards a squared Bessel process supported by a ray determined by the Perron vector of a matrix related to the branching mechanism of $\boldsymbol{X}$.

In Section 3, first we derive formulas of CLS estimators of the transformed parameters $\mathrm{e}^{\gamma+\kappa}$, $\mathrm{e}^{\gamma-\kappa}$ and $\int_0^1 \mathrm{e}^{s\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}s$, and then of the parameters $\gamma, \kappa$ and $\widetilde{\boldsymbol{\beta}}$. The reason for this parameter transformation is to reduce the minimization in the CLS