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Estimating the parameters of multiple chirp signals

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ABSTRACT

Chirp signals occur naturally in different areas of signal processing. Recently, Kundu and Nandi (2008) considered the least squares estimators of the unknown parameters of a chirp signal model and established their consistency and asymptotic normality properties. It is observed that the dispersion matrix of the asymptotic distribution of the least squares estimators is quite complicated. The aim of this paper is twofold. First, using a number theoretic result of Vinogradov (1954), we present a simplified form of the above mentioned dispersion matrix. Secondly, using the orthogonal structure of the different chirp components, we propose a step by step sequential estimation procedure of the unknown parameters of the model. Under the proposed sequential procedure, the problem of estimation of the parameters of a multiple chirp signal model reduces to solving only a two dimensional optimization problem at each step. It is observed that the estimators obtained by the proposed method are strongly consistent. Due to the complicated nature of the model, we could not establish the asymptotic distribution of the proposed sequential estimators. We perform some simulation experiments to compare the performance of the proposed and least squares estimators for small sample sizes, and for different parameter values. It is observed that the mean squared errors of the proposed estimators are very close to the corresponding mean squared errors of the least squares estimators. Two real data sets have been analyzed for illustrative purposes.

1. Introduction

In this paper we consider the following multiple chirp model;

$$y(n) = \sum_{k=1}^{p} \left\{ A_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) \right\} + X(n), \quad n = 1, \ldots, N. \tag{1}$$

Here $y(n)$ is the real valued signal observed at $n = 1, \ldots, N$; A_k^0, B_k^0 are amplitudes, and α_k^0 and β_k^0 are frequency and frequency rate, respectively for $k = 1, \ldots, p$. The additive error $\{X(n)\}$ is a sequence of stationary random variables with mean zero and finite fourth moment. The explicit assumptions on $X(n)$ will be provided later.

The model (1) is popularly known as the 'chirp signal model' in the signal processing literature, and occurs naturally in various areas of science and engineering, particularly in physics, sonar, radar and communications. Chirp signal models have been used extensively to measure the distance of a moving object, emitting chirp signal, from a fixed receiver. This

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phenomena is used by different animal species, like bats, whales, to detect their prey. Shrill voices of birds are also examples of chirp signal. In the class of periodic models wherein the frequency varies with time, chirp signal model is the simplest possible model. In other words, chirp signal has time dependent frequency or can incorporate frequency modulation, in electrical engineering terminology. Extensive work on chirp signal model, mainly when $p = 1$, has been carried out by several authors, see for example, Abatzoglou [1], Kumaresan and Verma [4], Djuric and Kay [2], Gini et al. [3], Nandi and Kundu [7], Kundu and Nandi [5] and the references cited therein.

Saha and Kay [10] first introduced the multiple chirp signal model (1), and provided the maximum likelihood estimators of the unknown parameters using importance sampling procedure under the assumptions that $X(n)$s are independent and identically distributed ($i.i.d.$) normal random variables. Kundu and Nandi [5] studied the properties of the least squares estimators (LSEs) of the model (1), and proved the strong consistency and asymptotic normality when $X(n)$s are obtained from a linear stationary process. However, the structure of the dispersion matrix of the asymptotic distribution of the LSEs in Kundu and Nandi [5], is quite complicated.

It is interesting to observe that the LSE of α_k^0 has the convergence rate $O_p(N^{-3/2})$, whereas the LSE of β_k^0 has the convergence rate $O_p(N^{-5/2})$. But it is also observed that, if $p \geq 2$, finding the LSEs is a numerically challenging problem. For the model (1), it involves solving a $2p$-dimensional optimization problem. Therefore, for large p, it becomes a highly computationally intensive procedure.

The aim of this paper is twofold. First, using a number theoretic result of Vinogradov [11] we provide a simplified structure of the dispersion matrix of the asymptotic distribution of the LSEs. The second aim of this paper is to provide an estimation procedure which is computationally less demanding and produces estimators of the unknown parameters, which behave in a manner very similar to the LSEs. If p is known, using the fact that the regressor vectors are orthogonal, we provide a step by step sequential estimation procedure for estimation of the amplitudes, frequencies and frequency rates. It is observed that $2p$-dimensional optimization procedure can be reduced to p sequential 2-dimensional (2-D) optimization problems. Therefore, if p is large, the proposed sequential procedure can be very effective.

It is observed that if p is not known, and we fit a lower order model, i.e. when the assumed number of components is less than the actual number of components, then the proposed estimators converge almost surely to the corresponding true parameter values. If we fit a higher order model, i.e. assumed number of components is more than the actual number of components, then the amplitude estimates obtained after the p-th step converge to zero almost surely.

Due to the complicated nature of the model, we could not establish the asymptotic distribution of the proposed sequential estimators. However, based on a conjecture in number theory, it can be shown that the asymptotic distributions of the LSEs and the proposed sequential estimators are the same. We perform some simulation experiments to study the behavior of the proposed estimators, and compare their performances with the LSEs. It is observed that the mean squared errors (MSEs) of the LSEs and sequential estimators are very close to each other. Finally, we provide the analysis of two real data sets for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we mainly state the model assumptions and preliminary results. Simplified form of the dispersion matrix of the asymptotic distribution of the LSEs is presented in Section 3. The sequential estimators are provided in Section 4. Numerical results and the analysis of two data sets are presented in Section 5, and finally we conclude the paper in Section 6. All the proofs are supplied in the Appendices.

2. Model assumptions and preliminary results

2.1. Model assumptions

We make the following assumptions on the error random variables.

Assumption 1. The error random variable $X(n)$ satisfies the following condition;

$$X(n) = \sum_{j=-\infty}^{\infty} a(j)e(n-j), \tag{2}$$

where $\{e(n)\}$ is a sequence of $i.i.d.$ random variables with mean zero,variance σ^2, finite fourth moment, and

$$\sum_{j=-\infty}^{\infty} |a(j)| < \infty. \tag{3}$$

It may be mentioned that Assumption 1 is a standard assumption for stationary linear process, and any finite dimensional stationary AR, MA or ARMA process can be represented as (2), when $a(j)$s satisfy (3).

We use the following notations. The parameter vector of the k-th component is represented as $\theta_k = (A_k, B_k, \alpha_k, \beta_k)$, the true parameter vector as $\theta_k^0 = (A_k^0, B_k^0, \alpha_k^0, \beta_k^0)$, for $k = 1, \ldots, p$, and the parameter space as $\Theta = [-M, M] \times [-M, M] \times [0, \pi] \times [0, \pi]$; where $M > 0$ is a real number.

Assumption 2. It is assumed that θ_k^0 is an interior point of Θ, α_k^0s are distinct, and β_k^0s are also distinct for $k = 1, \ldots, p$.