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A bivariate Gompertz–Makeham life distribution

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ABSTRACT

In the context of actuarial science, Gompertz (1825) utilized a differential equation to derive the life distribution that carries his name. Subsequently, De Morgan (1860), Woolhouse (1863), and Kaminsky (1983) derived the Gompertz distribution from functional equations. This paper focuses on bivariate versions of Kaminsky's functional equation. A limiting version yields the bivariate exponential distribution of Marshall and Olkin (1967).

1. Introduction

With applications to actuarial science in mind, Gompertz [7] set out to derive a distribution appropriate for describing human life lengths. To this end, he made an assumption regarding the degradation of a man's power to avoid death, formalized the assumption as a differential equation involving the hazard rate of the life distribution, and solved the equation to derive the distribution we now know as the Gompertz distribution.

To describe this distribution, some notation is required. For a random variable X, let $\bar{F}(x) = P\{X > x\}$ denote the *survival function* and let $R(x) = -\log \bar{F}(x)$ denote the *hazard function*. When F has a density f, the ratio $r(x) = f(x)/\bar{F}(x)$ is called the *hazard rate*. For the Gompertz distribution,

$$R(x|\lambda, \xi) = \xi(e^{\lambda x} - 1) \tag{1.1}$$

and

$$r(x|\lambda, \xi) = \partial R(x|\lambda, \xi)/\partial x = \xi\lambda e^{\lambda x}, \quad x \geq 0,\ \xi, \lambda > 0. \tag{1.2}$$

Note that with $\lambda\xi = c$ fixed,

$$\lim_{\lambda\to 0} R(x|\lambda, \xi) = \lim_{\lambda\to 0} \frac{c}{\lambda}\left(e^{\lambda x} - 1\right) = cx; \tag{1.3}$$

this is the hazard function of the exponential distribution. Because exponential distributions are limits of Gompertz distributions the Gompertz distribution can be regarded as a two-parameter extension of the exponential distribution, a property also of the Weibull and gamma distributions.

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Gompertz found that his distribution did not fit his data very well; adjustments were necessary. In his words, he recognized "that death may be the consequence of coexisting causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or increased inability to withstand destruction". Gompertz did not put these two causes together but this was done by Makeham [12]. Makeham found that by considering the distribution of $U = \min(X, S)$ where X has a Gompertz distribution and S has an exponential distribution, the fit to data was significantly enhanced. The distribution of U is now known as the Gompertz–Makeham distribution.

To obtain bivariate Gompertz–Makeham distributions, a first step is to focus on deriving bivariate Gompertz distributions, i.e., bivariate distributions with Gompertz marginal distributions. Because the exponential distribution is a limit of Gompertz distributions, bivariate exponential distributions can be obtained as limits of bivariate Gompertz distributions. By combining these results, bivariate Gompertz–Makeham distributions are easily constructed.

A simple direct way to derive a bivariate Gompertz distribution is to note that if U, V, and W are independent random variables having Gompertz distributions with the same scale parameter, then

$$X = \min(U, W), \qquad Y = \min(V, W), \tag{1.4}$$

has a bivariate Gompertz distribution (see Section 5). Similar constructions have been employed to obtain various bivariate distributions (see Section 8).

The approach in this paper is entirely different. Here the starting point is a condition that characterizes the desired marginal distributions. Two well-known characterizations are the memoryless property of the exponential distribution and the independence of the sample mean and variance of the normal distribution. Both of these characterizations suggest bivariate versions.

The Gompertz distribution has been derived by De Morgan [4], Woolhouse [19], and more recently by Kaminsky [10] using functional equations. These derivations are discussed in some detail by Marshall and Olkin [16]. Many ways of generating bivariate distributions with specified marginals are discussed extensively by Balakrishnan and Lai [2]. The methods include use of copulas [17] and the use of mixtures [15]. These methods are general and are not based on characteristics of the marginals.

The present focus is on a functional equation, due to Kaminsky [10], that characterizes the Gompertz distribution. Bivariate versions of the functional equation are used to derive bivariate Gompertz distributions. Limiting bivariate exponential distributions are obtained, and from these results, bivariate Gompertz–Makeham distributions are constructed.

2. A functional equation approach: introduction

The exponential distribution is characterized by the well-known "lack-of-memory" property

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} = \bar{F}(x), \quad x, t \geq 0, \tag{2.1}$$

or equivalently,

$$R(x+t) = R(x) + R(t). \tag{2.2}$$

A modification of this functional equation was proposed by Kaminsky [10]:

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} = \left[\bar{F}(x)\right]^{\phi(t)}, \quad x, t \geq 0, \tag{2.3}$$

or equivalently,

$$R(x+t) = \phi(t)R(x) + R(t). \tag{2.4}$$

Because of the relationship between (2.1) and (2.3), this paper is closely related to the authors' paper on multivariate exponential distributions [14] which investigates bivariate versions of (2.1).

Proposition 2.1 ([10]). *A univariate survival function $\bar{F}$ satisfies (2.3) for some function ϕ independent of x if and only if*

(i) *F is an exponential distribution and $\phi(t) \equiv 1$ or*
(ii) *F is a Gompertz distribution and $\phi(t) = e^{\lambda t}$ for some $\lambda > 0$ or*
(iii) *F is a negative Gompertz distribution and $\phi(t) = e^{-\lambda t}$ for some $\lambda > 0$.*

See Marshall and Olkin [16, p. 372].

For Eq. (2.1), Marshall and Olkin [14] considered two bivariate versions:

$$\frac{\bar{F}(x+s, y+t)}{\bar{F}(s, t)} = \bar{F}(x, y), \quad x, y, s, t \geq 0, \tag{2.5}$$

and

$$\frac{\bar{F}(x+t, y+t)}{\bar{F}(t, t)} = \bar{F}(x, y), \quad x, y, t \geq 0. \tag{2.6}$$