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# Prediction of stationary Gaussian random fields with incomplete quarterplane past

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ABSTRACT

Let $\{X_{m,n} : (m, n) \in \mathbf{Z}^2\}$ be a stationary Gaussian random field. Consider the problem of predicting $X_{0,0}$ based on the quarterplane $Q = \{(m, n) : m \geq 0, n \geq 0\} \setminus \{(0, 0)\}$, but with finitely many observations missing. Two solutions are presented. The first solution expresses the best predictor in terms of the moving average coefficients of $\{X_{m,n}\}$, under the assumption that the spectral density function has a strongly outer factorization. The second solution expresses the prediction error variance in terms of the autoregressive coefficients of $\{X_{m,n}\}$; it requires the reciprocal of the density function to have a strongly outer factorization, and relies on a modified duality argument. These solutions are extended by allowing the quarterplane past to be replaced with a much broader class of parameter sets. This enables the solution, for example, of the quarterplane interpolation problem.


## 1. Introduction

The basic prediction problem for a centered stationary Gaussian process $\{X_n : n \in \mathbf{Z}\}$ is to estimate $X_0$ based on the "past", namely, the closed linear span of $\{\dots, X_{-3}, X_{-2}, X_{-1}\}$, in the least-squares sense. The classic Szegő–Kolmogorov–Wiener solution for the prediction error relies on the Hilbert space isomorphism induced by the mapping $X_n \longmapsto e^{-in\theta}$ from the span of $\{X_n : n \in \mathbf{Z}\}$ onto the function space $L^2(\mu)$, where $\mu$ is the spectral measure of the process. The solution then invokes deep results from the theory of functions on the unit circle $\mathbf{T}$ of the complex plane.

This basic prediction problem has been extended in a number of natural directions, again drawing upon function theory on $\mathbf{T}$. These extensions include prediction $n$ steps ahead [29], and prediction with certain observations added to or deleted from the past [28]. The practical value of these cases is clear.

These issues also extend to the setting of random fields. If $\{X_{m,n} : (m, n) \in \mathbf{Z}^2\}$ is a centered stationary Gaussian random field, then the analogous basic prediction problem is to estimate $X_{0,0}$ based on the span of $\{X_{m,n} : (m, n) \in S\}$, where $S$ is a subset of $\mathbf{Z}^2$ playing the role of the past. In this situation, the random variables $X_{m,n}$ may represent observations that are spatially distributed over an integer lattice, rather than in time. Thus, in contrast with the time series case, there is not a canonical choice of past parameter set $S$. Indeed, each specific application involving observations indexed by lattice points of $\mathbf{Z}^2$ may drive a particular choice of $S$. However, from a theoretical standpoint, a very important special case arises when the past is generated by a quarterplane $Q$, defined by

$$Q = \{(m, n) : m \geq 0, n \geq 0\} \setminus \{(0, 0)\}. \tag{1}$$

In this situation, the trigonometric isomorphism $X_{m,n} \longmapsto e^{ims+int}$ then brings in the theory of functions on $\mathbf{T}^2$. For analytical reasons, the choice of quarterplane past enables us to investigate the prediction problem by analogy with the one-parameter case, by drawing upon the rich function theory on $\mathbf{T}^2$.

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In [34], the basic quarterplane prediction problem is solved when the stationary Gaussian random field possesses a "one-sided moving average representation", in a certain strict sense. Building on this foundation, Kohli and Pourahmadi [22] studied some related extensions. They obtained the best predictor of $X_{0,0}$ based on the quarterplane $Q$ with finitely many observations added. Their solution expresses the error variance formulas in terms of the moving average (MA) parameters of the random field. They found that expressing the solution in terms of the autoregressive (AR) parameters is somehow connected to the "edge effects" associated with spectral density estimation (see [17]). In addition, they outlined an approach to the quarterplane prediction problem with finitely many missing observations, taking note of several open technical issues.

The present work is concerned with solving the quarterplane prediction problem with finitely many missing values, which was left open in [22]. The methods used here sidestep, but do not solve, the technical obstacles noted in [22]. Two solutions are obtained. The first solution, representing a direct approach, expresses the best predictor in terms of the MA coefficients of the random field. This solution requires further development of the underlying function theory, having to do with factorization of the spectral density function. The results include formulas for the best predictor and the prediction error variance which are informative and straightforward to use. These results are of immediate value to applications in which a finite number of observations from the quarterplane past are missing or corrupted.

The second solution requires more stringent spectral conditions, and employs a modified duality argument to express the prediction error variance in terms of the AR coefficients of the random field. It is thereby shown that the prediction problem with incomplete past is equivalent to a related prediction problem with added observations. Again, the resulting formula for the prediction error variance is explicit and tractable. In addition, these results can be used in applications to gauge the relative value of any particular observation from a quarterplane past.

Both approaches employ a central idea from [22] to derive the best predictor of $X_{0,0}$ in two steps; the first step is to solve a related standard prediction problem; the second step is to adjust the prediction according to the added or missing values. In the present setting these steps are made possible by a number of contributions to function theory. Properties and examples of "strongly outer functions" on the polydiscs are derived, which pertain to the spectral factorization. Furthermore, it is shown that the associated methods extend in a straightforward way when the quarterplane past of the random field is replaced by a much broader class of subsets of $\mathbf{Z}^2$, which we call $Q$-invariant. This enables us to solve, for example, the related quarterplane interpolation problem. Thus, while the quarterplane $Q$ might at first appear to be an artificial choice of past with limited practical value, it turns out to be the key to solving a much wider set of cases.

For further reference, the prediction of a stationary one-parameter process, with the past altered by finitely many added or missing values, is the subject of [1,2,28,29,31]. The results of this paper, along with those of [22], treat the analogous two-parameter case where the past is a quarterplane. The corresponding problems with infinite-variance processes are treated in [3,8,21,25,27,32], and multivariate Gaussian processes in [9,30]. With respect to the two-parameter case, other choices of the past $S$ have been investigated. The papers [18,19], for instance, are concerned with subsets $S$ such that (i) $(0, 0) \notin S$, (ii) $S \cup (0, 0)$ is an additive semigroup, and (iii) $\mathbf{Z}^2$ is the disjoint union $S \cup \{(0, 0)\} \cup -S$. These parameter sets are called "halfplanes". In the works [4,23], the past is determined by a lexicographical ordering of $\mathbf{Z}^2$; they are special cases of halfplanes. The algebraic properties of these halfplanes have deep analytical consequences, which make possible the solution of the corresponding prediction problems. Yet other approaches to the prediction of random fields have been taken; see, for example, [7,11,12,15,20,26]. Duality type arguments in prediction theory appear in [8,16,27–29,35]; see also [21] for a very unusual implementation of this concept. The methods of this paper depend on the theory of functions of two variables. Our reference for function theory in one variable is [14], particularly Chapters 2, 6 and 7. For the background in function theory of polydiscs, see [33]. The results of [34] serve as a launching point for the basic quarterplane prediction problem. The related notion of a strongly outer function, introduced and explored in [5,6,10], is further developed here. The papers [13,24] contain yet more contributions to the associated functional analysis.

This paper is organized as follows. The next section sets forth the notation and assumptions of this paper. In particular, the quarterplane MA and AR representations of the random field $\{X_{m,n} : (m, n) \in \mathbf{Z}^2\}$ are presented, and their relationship to the spectral factorization is reviewed. The related background in function theory is then presented. This includes the definition, properties and examples of strongly outer functions in the Hardy classes of the circle and torus, and their connection to the prediction problem. The concept of $Q$-invariance is introduced. New results for strongly outer functions are obtained in Section 3. They include large classes of examples, and necessary tools for solving the prediction problem. Section 4 contains the first solution to the prediction problem with incomplete quarterplane past, expressing the best prediction in terms of the MA coefficients of the random field. In Section 5 the second solution to the prediction problem is obtained, by using a modified duality method. It expresses the prediction error variance in terms of the AR coefficients of the random field. Section 6 argues that many of the methods of the previous sections apply when the quarterplane past $Q$ is replaced by a parameter set that is $Q$-invariant, a very broad expansion in scope. The basic prediction problem for $Q$-invariant past is solved, and the corresponding duality result is obtained. As an application of the latter, the quarterplane interpolation problem is solved.

## 2. Preliminaries

This section establishes the notation and assumptions for this project, and furnishes an overview of the needed function theoretic background in one and two variables.

Throughout this paper, $Q$ is the quarterplane defined in (1). Let $\lambda$ be normalized Lebesgue measure on the unit circle $\mathbf{T}$ of the complex plane. Throughout this paper we take $\{X_{m,n} : (m, n) \in \mathbf{Z}^2\}$ to be a complex-valued, centered, stationary