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On mixtures of copulas and mixing coefficients

Martial Longla

Department of Mathematics, University of Mississippi, 38677 University, MS, United States

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ABSTRACT

We show that if the density of the absolutely continuous part of a copula is bounded away from zero on a set of Lebesgue measure 1, then that copula generates "lower ψ-mixing" stationary Markov chains. This conclusion implies ϕ-mixing, ρ-mixing, β-mixing and "interlaced ρ-mixing". We also provide some new results on the mixing structure of Markov chains generated by mixtures of copulas.

1. Introduction

1.1. Motivation and background

The importance of mixing coefficients in the theory of central limit theorems for weakly dependent random sequences is an established fact. So, it is important to understand the mixing structure of various models. In the recent years, many researchers have been trying to provide sufficient conditions for mixing at various rates. These efforts have invited many people to investigate the properties of copulas, because they capture the dependence structure of stationary Markov chains. Mixtures of distributions are very popular in modeling.

We present here a review of mixing coefficients and their convergence for mixtures of copulas. Convergence of mixing coefficients allows various central limit theorems for estimation of functions of the random variables or inference on model parameters.

We have initially shown in [6] a result on mixtures of copulas for absolute regularity. A second result on ρ-mixing for mixtures of copulas was provided in [4]. Here, we extend the results to other mixing coefficients that are not less important.

We are providing here an improvement of one of our previous results. Namely, we have shown in [6], that Markov chains generated by a copula are ϕ-mixing, when the density of its absolutely continuous part is bounded away from 0. This paper provides the proof that under this condition the Markov chains are lower ψ-mixing. Taking into account Theorems 1.2 and 1.3 of Bradley [2], this conclusion implies "interlaced ρ-mixing".

E-mail address: mlongla@olemiss.edu.

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1.2. Definitions

Suppose $(\Omega, \mathscr{F}, P)$ is a probability space, and $(X_m, m \in \mathbb{N})$ is a stationary Markov chain generated by a copula $C(x, y)$ on this space. Recall that a copula is a joint cumulative distribution function on $[0, 1]^2$ with uniform marginals; and a copula-based Markov chain is nothing but a stationary Markov chain represented the copula of its consecutive states and its invariant distribution. Some examples of famous copulas are $\Pi(x, y) = xy$—the independence copula, $W(x, y) = \max(x+y-1, 0)$—the Hoeffding lower bound and $M(x, y) = \min(x, y)$—the Hoeffding upper bound. Π is the copula of independent random variables, W induces on $[0, 1]^2$ a singular probability measure with support $S_1 = \{(x, y) \in [0, 1]^2 : x + y = 1\}$ and M induces a singular probability measure with support $S_2 = \{(x, y) \in [0, 1]^2 : x = y\}$. The copula $C(x, y)$ induces on the unit square a probability measure that we will denote P_C. This probability measure acts on sets $A = (0, x] \times (0, y]$ by $P_C(A) = C(x, y)$. For a set of copulas $C_1(x, y), \ldots, C_k(x, y)$ and some strictly positive numbers $a_1, \ldots, a_k$ such that $\sum_{i=1}^k a_i = 1$, $C(x, y) = \sum_{i=1}^k a_i C_i(x, y)$ is a copula and the measure it induces on $[0, 1]^2$ is $P_C = \sum_{i=1}^k a_i P_{C_i}$. Let also P_{CF} denote the probability measure induced by $C(x, y)$ and the univariate cumulative distribution function F on our probability space. Let $f_{,i}(x, y)$ denote the derivative with respect to the ith variable of the function f at the point (x, y). The fold product of the copulas $C_1(x, y)$ and $C_2(x, y)$ is the copula defined by

$$C(x, y) = C_1 * C_2(x, y) = \int_0^1 C_{1,2}(x, t)C_{2,1}(t, y)dt.$$

This operation is associative and distributive over convex combinations of copulas. The widely used notation of the powers of $C(x, y)$ is $C^m(x, y) = C^{m-1} * C(x, y)$ with $C^1(x, y) = C(x, y)$ is called mth fold product of $C(x, y)$. This copula represents that of the random vector (X_0, X_m), where $(X_0, X_1, \ldots, X_m)$ is a stationary Markov chain with copula $C(x, y)$ and the uniform marginal.

Note that for any vector random variable (U, V) with joint distribution $H(u, v)$ and respective marginals F_U and F_V, $H(u, v) = C(F_U(u), F_V(v))$ for some copula $C(x, y)$, where F_X represents the distribution of the random variable X. Sklar's theorem ensures uniqueness of this representation for continuous random variables. Let $c(x, y)$ be the density of the absolutely continuous part of the copula $C(x, y)$ and $c_m(x, y)$ the density of the absolutely continuous part of the mth fold product of $C(x, y)$. For definitions, check Longla and Peligrad [6] or Longla [4].

Using the Lebesgue measure as invariant distribution after rescaling the effect of the marginal distribution, taking into account the notations from [6] the mixing coefficients of interest for an absolutely continuous copula and an absolutely continuous invariant distribution for the states of the stationary Markov chain that they generate are defined below.

Definition 1. Let $\mathscr{B}$ denote the Borel σ-algebra on $[0, 1]$. Let λ be the Lebesgue measure on $[0, 1]$, B^c be the complement of the set B in $[0, 1]$. $\mathbb{E}_\mu(f)$ is the expected value of the random variable $f(X)$ under the probability measure μ and $\|f\|_2 = \left(\int_0^1 f^2(x)dx\right)^{1/2}$.

$\rho_n = \sup_{f,g}\left\{\int_0^1\int_0^1 c_n(x, y)f(x)g(y)dxdy, \int_0^1 f(x)dx = 0, \int_0^1 g(x)dx = 0, \|f\|_2 = \|g\|_2 = 1\right\}$ is the maximal coefficient of correlation. The Markov chain is ρ-mixing, if ρ_n converges to 0.

$\phi_n = \sup_{B\subset\mathscr{B}\cap[0,1]} ess\sup_{x\in[0,1]} |\int_B (c_n(x, y) - 1)dy|$ is the uniform mixing coefficient. The Markov chain is uniformly mixing, if ϕ_n converges to 0.

$\beta_n = \int_0^1 \sup_{B\in\mathscr{B}\cap[0,1]} |\int_B (c_n(x, y) - 1)dy|dx$ is the absolute regularity mixing coefficient. The Markov chain is absolutely regular, if β_n converges to 0. If the convergence of β_n is exponential, then we say that the Markov chain is geometrically ergodic.

$\psi'_n = \inf_{A,B\in\mathscr{B},\lambda(A)\lambda(B)>0} \frac{\int_A\int_B c_n(x,y)dxdy}{\lambda(A)\lambda(B)}$ is the "lower ψ-mixing". We say that the Markov chain is "lower ψ-mixing" or ψ'-mixing, if ψ'_n converges to 1.

$\psi_n = \sup_{A,B\in\mathscr{B},\lambda(A)\lambda(B)>0} \frac{|\int_A\int_B (c_n(x,y)-1)dxdy|}{\lambda(A)\lambda(B)}$ is the ψ-mixing coefficient. The Markov chain is ψ-mixing if ψ_n converges to 0.

These coefficients have a more complex formulation for a general random sequence. For the sake of clarity, we shall provide the general definitions of these coefficients for stationary Markov chains with marginal F. Let $\mathscr{R} = \sigma(X_i)$ be the σ-algebra generated by X_i and $\mathbb{L}_2(\sigma(X_i))$ be the space of random variables that are $\sigma(X_i)$ measurable and square integrable. Let $Corr(X, Y)$ be the correlation coefficient of the random variables X and Y. The derivation of the above equivalent forms is provided in [6].

Definition 2. Under the above assumptions and notations, let $\mathscr{R}^2 = \sigma(X_0, X_n)$, where $\sigma(X_m, m \in S)$ is the σ-algebra generated by the random variables indexed by S. For every positive integer n, let μ_n be the measure induced by the distribution of (X_0, X_n). Let μ be the measure induced by the distribution of X_0. The coefficients of interest are defined as follows:

$$\rho_n = \rho(\sigma(X_0), \sigma(X_n)) := \sup\{Corr(f, g), f \in \mathbb{L}_2(\sigma(X_0)), g \in \mathbb{L}_2(\sigma(X_n))\},$$
$$\phi_n = \phi(\sigma(X_0), \sigma(X_n)) := \sup_{A,B\in\mathscr{R},\mu(A)>0} |P(X_n \in B|X_0 \in A) - \mu(B)|,$$