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Distributions on matrix moment spaces

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ABSTRACT

In this paper we define distributions on the moment spaces corresponding to $p \times p$ real or complex matrix measures on the real line with an unbounded support. For random vectors on the unbounded matricial moment spaces we prove the convergence in distribution to the Gaussian orthogonal ensemble or the Gaussian unitary ensemble, respectively.

1. Introduction

In recent years there has been considerable interest in generalizing many of the results on classical moment theory to the case of matrix measures. Wiener and Masani [26] introduced matrix measures on the unit circle in the study of multivariate stochastic processes and their spectral theory. Whittle [25] followed the same approach and established a connection to matrix polynomials, that is, polynomials with matricial coefficients. Already Karlin and McGregor [17] studied a random walk with a doubly infinite transition matrix with help of matrix polynomials, however without special consideration of the matricial structure. Delsarte et al. [3] orthogonalized polynomials with respect to matrix measures on the unit circle. Duran and van Assche [11], Duran [8,9] and Duran and Lopez-Rodriguez [10] were the first who investigated matrix orthogonal polynomials with respect to matrix measures on the real line and generalized many results from the scalar case to the matrix case. Typical examples include the three-term-recursion, quadrature formulas and ratio asymptotics. Applications to stochastic processes with two-dimensional state space were discussed by Dette et al. [6], who expressed transition probabilities and the recurrence of states in terms of matrix measures and matrix orthogonal polynomials.

In contrast to moment spaces corresponding to (probability) measures the structure of moment spaces corresponding to matrix measures is much richer and not very well understood. In the scalar case Chang et al. [2] investigated a uniform distribution on the moment space corresponding to measures on the interval [0, 1]. Their investigation was motivated by the consideration of a "typical" point in the moment space and they studied the asymptotic properties of random moment

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vectors with increasing dimension. Gamboa and Lozada-Chang [12] considered large deviation principles for random moment sequences on this space, while Lozada-Chang [18] investigated similar problems for moment spaces corresponding to more general functions defined on a bounded set. More recently, Gamboa and Rouault [14] discussed random spectral measures related to moment spaces of measures on the interval [0, 1] and moment spaces related to measures defined on the unit circle. Dette and Nagel [4] considered distributions on moment spaces corresponding to scalar measures on the real line with an unbounded support.

For matrix measures the corresponding moment of a matrix measure is given by a symmetric (Hermitian) matrix and Dette and Studden [7] obtained a characterization of the compact moment space corresponding to matrix measures on a compact interval. Dette and Nagel [5] used these results to investigate the asymptotic properties of random vectors with values in the moment space corresponding to matrix measures on the interval [0, 1]. The aim of the present paper is to get a better understanding of the properties in the non compact case. For this purpose we define probability distributions on matrix moment spaces corresponding to measures with an unbounded support and study their asymptotic behavior with an increasing dimension.

The remaining part of this paper is organized as follows. In Section 2 we introduce the basic notation, define distributions on the moment spaces corresponding to matrix measures on unbounded intervals and state our main results. In Section 3 we consider matrix orthogonal polynomials and their relation to moments of matrix measures. In Section 4 we use this relation to prove our main results. Finally in Section 5 we extend these results to matrix moment spaces corresponding to matrix measures with complex entries. Finally some technical details have been deferred to the Appendix.

2. Matrix moment spaces

Throughout this paper let $(\mathcal{S}_p(\mathbb{R}), \mathcal{B}(\mathcal{S}_p(\mathbb{R})))$ denote the measurable space of all $p \times p$ symmetric matrices with real entries, where $\mathcal{B}(\mathcal{S}_p(\mathbb{R}))$ is the Borel field corresponding to the Frobenius norm $\|A\| = \sqrt{\operatorname{tr}(A^2)}$ on $\mathcal{S}_p(\mathbb{R})$. For properties of this norm and general results in matrix theory we refer to the book of Horn and Johnson [16]. The set $\mathcal{S}_p^+(\mathbb{R}) \subset \mathcal{S}_p(\mathbb{R})$ denotes the subset of positive definite matrices and for a matrix $A \in \mathcal{S}_p(\mathbb{R})$, $|A|$ is the determinant of A. Let T be a subset of the real line with corresponding Borel field $\mathcal{B}(T)$. A ($\mathcal{S}_p(\mathbb{R})$-valued) matrix measure Σ on a measurable space $(T, \mathcal{B}(T))$ is a $p \times p$ matrix of signed measures on $(T, \mathcal{B}(T))$ such that for all Borel sets $A \subset T$ the matrix $\Sigma(A)$ is symmetric and nonnegative definite. Additionally we require the matrix measure to be normalized, that is $\Sigma(T) = I_p$, where I_p denotes the $p \times p$ identity matrix. We consider on $\mathcal{S}_p(\mathbb{R})$ the integration operator

$$dX := \prod_{i \le j} dx_{ij}, \tag{2.1}$$

the product Lebesgue measure with respect to the independent entries of a symmetric matrix. For an integrable function $f : \mathcal{S}_p(\mathbb{R}) \to \mathbb{R}$ the integral

$$\int f(X)dX \tag{2.2}$$

is the iterated integral with respect to each of the elements $x_{ij},\ i \le j$ (see Muirhead [20] or Gupta and Nagar [15]). The kth moment of a matrix measure is then defined as

$$M_k(\Sigma) := \int x^k d\Sigma(x) \tag{2.3}$$

for $k \ge 0$. The set of all $\mathbb{R}$-valued matrix measures on $(T, \mathcal{B}(T))$ for which all moments exist is denoted by $\mathcal{P}_p(T)$ and we define the nth moment space of matrix measures by

$$\mathcal{M}_{p,n}(T) := \left\{ (M_1(\Sigma), \dots, M_n(\Sigma))^T \,|\, \Sigma \in \mathcal{P}_p(T) \right\}. \tag{2.4}$$

Analogous to the compact case in Dette and Nagel [5] we obtain a characterization of the moment spaces $\mathcal{M}_{p,n}([0,\infty))$ and $\mathcal{M}_{p,n}(\mathbb{R})$ in terms of Hankel matrices, which are defined for matrices $M_k \in \mathcal{S}_p(\mathbb{R})$, $k \ge 0$ as

$$\underline{H}_{2m} = \begin{pmatrix} M_0 & \cdots & M_m \\ \vdots & & \vdots \\ M_m & \cdots & M_{2m} \end{pmatrix}, \qquad \overline{H}_{2m} = \begin{pmatrix} M_1 - M_2 & \cdots & M_m - M_{m+1} \\ \vdots & & \vdots \\ M_m - M_{m+1} & \cdots & M_{2m-1} - M_{2m} \end{pmatrix}, \tag{2.5}$$

and

$$\underline{H}_{2m+1} = \begin{pmatrix} M_1 & \cdots & M_{m+1} \\ \vdots & & \vdots \\ M_{m+1} & \cdots & M_{2m+1} \end{pmatrix}, \qquad \overline{H}_{2m+1} = \begin{pmatrix} M_0 - M_1 & \cdots & M_m - M_{m+1} \\ \vdots & & \vdots \\ M_m - M_{m+1} & \cdots & M_{2m} - M_{2m+1} \end{pmatrix}. \tag{2.6}$$

The following lemmas give a characterization of $\mathcal{M}_{p,n}([0,\infty))$ and $\mathcal{M}_{p,n}(\mathbb{R})$. The proof follows by similar arguments as in Dette and Studden [7] and is therefore omitted. Note that the authors consider non-normalized measures, but the arguments can be extended to matrix probability measures.