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The joint distribution of Studentized residuals under elliptical distributions

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ABSTRACT

Scaled and Studentized statistics are encountered frequently, and they often play a decisive role in statistical inference and testing. For instance, taking the sample mean vector $\bar{\boldsymbol{X}} = \sum_{j=1}^{N} \boldsymbol{X}_j/N$ and the sample covariance matrix $S = \sum_{j=1}^{N}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})(\boldsymbol{X}_j - \bar{\boldsymbol{X}})'/(N-1)$ for an iid sample $\{\boldsymbol{X}_j\}_{j=1}^{N}$, some statistics for testing normality of the underlying distribution consist of the scaled residuals (the Studentized residuals or the transformed samples), $\boldsymbol{Y}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})$ $(j = 1, 2, \ldots, N)$. In this paper, the distribution of the random matrix the columns of which consist of the scaled residuals is derived under elliptical distributions. Also exact distributions of Studentized statistics are discussed as an application of the main result.

1. Introduction

Let $\boldsymbol{X}$ be a p-dimensional random vector distributed according to an *elliptical contoured distribution* (or briefly, *elliptical distribution*) which has a probability density function (pdf) of the form

$$f(\boldsymbol{x}) = K_p|\Lambda|^{-1/2}g((\boldsymbol{x}-\boldsymbol{\mu})'\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})), \tag{1}$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$, Λ is a positive definite matrix of order p, g is a real valued function, and K_p is a normalizing constant (see, for example, [8, Section 2.6.5], [12, Section 1.5]). Consequently, the characteristic function (cf) of $\boldsymbol{X}$ can be expressed as

$$\Psi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\Lambda\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^p,\ i = \sqrt{-1}. \tag{2}$$

If they exist, $\mathrm{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ and $\Sigma = \mathrm{Cov}[\boldsymbol{X}] = -2\psi'(0)\Lambda \equiv c\Lambda > 0$.

Let $\{\boldsymbol{X}_j\}_{j=1}^{N}$ be an iid random sample, and let the sample mean vector and the sample covariance matrix be

$$\bar{\boldsymbol{X}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{X}_j, \tag{3}$$

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$$S = \frac{1}{n}\sum_{j=1}^{N}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})(\boldsymbol{X}_j - \bar{\boldsymbol{X}})', \quad n = N - 1 \geq p, \tag{4}$$

respectively.

Mardia [11] has proposed measures of multivariate skewness and kurtosis, which are defined by

$$b_{1,p} = N\sum_{j,k=1}^{N}\left[(\boldsymbol{X}_j - \bar{\boldsymbol{X}})'(nS)^{-1}(\boldsymbol{X}_k - \bar{\boldsymbol{X}})\right]^3, \tag{5}$$

$$b_{2,p} = N\sum_{j=1}^{N}\left[(\boldsymbol{X}_j - \bar{\boldsymbol{X}})'(nS)^{-1}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})\right]^2. \tag{6}$$

These can be used to test whether N observations $\boldsymbol{X}_j$ are drawn from a normal population, and many authors (see, for example, [16]) have discussed applications of (5) and (6) to test for sphericity and elliptical symmetry. Characteristic of the statistics (5) and (6) and many similar quantities is that both of them consist of transformed samples called the *scaled residuals* (see [15]) or the *Studentized residuals* (see [14]),

$$\boldsymbol{W}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}}), \quad j = 1, \ldots, N, \tag{7}$$

where $S^{-1/2}$ denotes the inverse of a symmetric square root matrix of S (hereafter, we denote a symmetric square root of a $p \times p$ matrix A by $A^{1/2}$). As Fang and Liang [7] and Batsidis and Zografos [2] have mentioned, the transformed samples $\boldsymbol{W}_j, j = 1, \ldots, N$, are not independent, and $\boldsymbol{W}_j$'s distribution does not coincide with one of $\Sigma^{-1/2}(\boldsymbol{X}_j - \boldsymbol{\mu})$. However, if we would be able to obtain the exact joint distribution of $\boldsymbol{W}_j, j = 1, \ldots, N$, we could obtain a useful and powerful tool for testing not only elliptical symmetry but also various other statistical hypotheses.

Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be iid sample drawn from $\mathrm{EC}_p(\boldsymbol{0}, I_p)$, a *spherical distribution*. Recently, Iwashita et al. [9] have considered the sampling distribution of *Student's t-type statistics*

$$T_{\boldsymbol{\alpha}}(\boldsymbol{Z}) = \frac{(p-1)^{1/2}\boldsymbol{\alpha}'\boldsymbol{Z}}{\sqrt{\|\boldsymbol{Z}\|^2 - (\boldsymbol{\alpha}'\boldsymbol{Z})^2}}, \quad \|\boldsymbol{Z}\| = \left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{1/2}, \ \boldsymbol{\alpha}'\boldsymbol{\alpha} = 1, \ \boldsymbol{\alpha} \in \mathbb{R}^p \tag{8}$$

where

$$\boldsymbol{Z} = \sqrt{N}S^{-1/2}\bar{\boldsymbol{X}}, \tag{9}$$

the *Studentized sample mean vector*, and they have obtained the asymptotic expansion of the distribution of $T_{\boldsymbol{\alpha}}(\boldsymbol{Z})$ as

$$\Pr[T_{\boldsymbol{\alpha}}(\boldsymbol{Z}) \in B] = \int_B t_{p-1}(x)dx + o(N^{-2}), \tag{10}$$

which holds uniformly over all Borel subsets B of $\mathbb{R}$, where $t_{p-1}(x)$ denotes the density of the t-distribution with $p-1$ degrees of freedom. In addition, they have conducted some numerical experiments for typical $\mathrm{EC}_p(\boldsymbol{0}, I_p)$ to investigate the accuracy of (10), and affirmed that the asymptotic expansion yields a high degree of precision.

It is a well-known fact that if $\boldsymbol{X} \sim \mathrm{EC}_p(\boldsymbol{0}, I_p)$, then

$$T_{\boldsymbol{\alpha}}(\boldsymbol{X}) = \frac{(p-1)^{1/2}\boldsymbol{\alpha}'\boldsymbol{X}}{\sqrt{\|\boldsymbol{X}\|^2 - (\boldsymbol{\alpha}'\boldsymbol{X})^2}} \tag{11}$$

has a t_{p-1} distribution (see [8, Theorem 2.5.8] and [12, Theorem 1.5.7]). In a similar vein, we expect that the statistic (8) is exactly distributed as t_{p-1}, and, furthermore, (9) is exactly $\mathrm{EC}_p(\boldsymbol{0}, I_p)$.

In this paper, we aim to show that the distribution of the $p \times N$ random matrix W

$$W = [\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_N], \qquad \boldsymbol{W}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}}), \quad j = 1, \ldots, N, \tag{12}$$

has a *left-spherical distribution*, i.e., for every $p \times p$ orthogonal matrix H, HW has the same distribution as W (see [8, Definition 3.1.1]) when $\{\boldsymbol{X}_j\}_{j=1}^N$ are independently drawn from $\mathrm{EC}_p(\boldsymbol{0}, \Lambda)$ with pdf having the form (1), and that the Studentized sample mean vector (9) is exactly distributed as $\mathrm{EC}_p(\boldsymbol{0}, I_p)$.

In what follows, we write $\boldsymbol{X} \sim \mathrm{EC}_p(\boldsymbol{\mu}, \Lambda)$, when a random vector $\boldsymbol{X}$ has an elliptical distribution. Also, if we write $X \sim \mathrm{LS}_{p\times N}(\phi_X)$, then a $p \times N$ random matrix X has a left-spherical distribution with characteristic function $\phi_X(T'T)$, where T is a $p \times N$ matrix (see [8, Definitions 3.1.2 and 3.1.3]). Furthermore, if random matrices or random vectors X and Y have the same distribution, we write $X \stackrel{d}{=} Y$. Note that Fang and Zhang [8, Chapter III] considered random matrices with dimension $N \times p$ throughout. However, for convenience, we use "$p \times N$-random matrices" instead.

In Section 2, we introduce an important proposition, and apply it to obtain a basic property of spherical distributions which is crucial in the sequel. In Section 3, we show that the Studentized sample mean vector $\boldsymbol{Z}$ has an $\mathrm{EC}_p(\boldsymbol{0}, I_p)$ distribution, and the random matrix W is $\mathrm{LS}_{p\times N}(\phi_W)$, by making use of the result in Section 2 and with the help of theory concerned with properties of random matrices which are distributed over $\mathcal{O}(p)$, the set of orthogonal $p \times p$ matrices. In the last section, we comment on an application of our results obtained in the previous sections.