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On the Lorenz ordering of order statistics from exponential populations and some applications[☆]

Gaofeng Da [a], Maochao Xu [b,*], N. Balakrishnan [c,d]

[a] Department of Statistics and Finance, School of Management, University of Science and Technology of China, Hefei, Anhui, China
[b] Department of Mathematics, Illinois State University, Normal, IL, USA
[c] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada
[d] Department of Statistics, King Abdulaziz University, Jeddah, Saudi Arabia

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ABSTRACT

In this paper, the variability of order statistics from heterogeneous random samples is studied. It is shown that, *without any restriction on the parameters*, the variability of order statistics from heterogeneous exponential samples is always larger than that from homogeneous exponential samples in the sense of Lorenz ordering. Finally, some applications to reliability analysis and auction theory are pointed out.

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1. Introduction

Order statistics have received considerable attention in the literature since they play an important role in many areas including reliability, data analysis, goodness-of-fit tests, statistical inference, outliers, robustness, and quality control. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics arising from random variables $X_1, X_2, \ldots, X_n$. In the reliability context, $X_{n-k+1:n}$ denotes the lifetime of a k-out-of-n system. In particular, the parallel and series systems are 1-out-of-n and n-out-of-n systems, respectively. A lot of work on order statistics has been done in the case when the underlying variables are independent and identically distributed (i.i.d.); see [12,7,8] for more details. Studies of order statistics from heterogeneous samples began in early 1970s, motivated by robustness issues. After that, a lot of work has been done on order statistics from single-outlier and multiple-outlier models. Balakrishnan [6] synthesized developments on order statistics arising from independent and non-identically distributed random variables. One may also refer to [15,25] for some reviews on various recent developments.

The variability of order statistics has been studied by a number of authors including David and Groeneveld [11] and Arnold and Villaseñor [5]. Let $X_1, \ldots, X_n$ be independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$, respectively,

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* Corresponding author.
E-mail addresses: dagf@ustc.edu.cn (G. Da), mxu2@ilstu.edu (M. Xu), bala@mcmaster.ca (N. Balakrishnan).

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and $Y_1, \dots, Y_n$ be i.i.d. exponential random variables with common hazard rate λ. Let $X_{i:n}$ and $Y_{i:n}$, for $i = 1, \dots, n$, denote the corresponding sets of order statistics. Then, intuitively, $X_{i:n}$ should exhibit more variability than $Y_{i:n}$. This has been partially confirmed in the literature. For example, Sathe [23] proved that if $\lambda = \bar{\lambda}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$, then

$$\mathrm{Var}(X_{k:n}) \geq \mathrm{Var}(Y_{k:n}).$$

This result has been partially improved by Khaledi and Kochar [13] that if $\lambda = \hat{\lambda} = \left(\prod_{i=1}^{n} \lambda_i\right)^{1/n}$, then

$$\mathrm{Var}(X_{n:n}) \geq \mathrm{Var}(Y_{n:n}). \tag{1.1}$$

More recently, Kochar and Xu [16] further improved this result by showing that (1.1) holds if $\lambda \geq \lambda^*$, where

$$\lambda^* = \sum_{i=1}^{n} \frac{1}{i} \left\{ \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{1}{\sum_{j=1}^{k} \lambda_{i_j}} \right\}^{-1}.$$

It needs to be noted that $\lambda^* \leq \hat{\lambda} \leq \bar{\lambda}$ [16]. It can be seen that all existing results in the literature require certain conditions on the parameters for the comparison of variabilities of order statistics from heterogeneous and homogeneous samples. It turns out that Lorenz ordering is quite suitable for the purpose of this comparison; see, for example, [2,5,18]. For order statistics from the same exponential distribution, Arnold and Nagaraja [4] showed that, for $i \leq j$,

$$(n - i + 1)\mathrm{E}(Y_{i:n}) \leq (m - j + 1)\mathrm{E}(Y_{j:m}) \Longleftrightarrow Y_{j:m} \leq_{\mathrm{Lorenz}} Y_{i:n}, \tag{1.2}$$

where $\leq_{\mathrm{Lorenz}}$ means the Lorenz order defined formally in Section 2. Kochar and Xu [16] showed further that $X_{n:n} \geq_{\mathrm{Lorenz}} Y_{n:n}$ without any restriction on the parameter λ. This result states that the variability of largest order statistic from heterogeneous exponential samples is larger than the one from homogeneous exponential samples in the sense of Lorenz ordering. As a direct consequence of this result, the coefficient of variation (defined in Section 2) $\gamma_{X_{n:n}}$ has the following lower bound:

$$\gamma_{X_{n:n}} \geq \sqrt{\sum_{i=1}^{n} \frac{1}{i^2}} \Big/ \sum_{i=1}^{n} \frac{1}{i}.$$

However, due to the complicated form of distributions of order statistics from independent and non-identical variables, an analogous result for general order statistics has remained as an open problem. In this paper, we solve this problem by showing precisely that

$$X_{k:n} \geq_{\mathrm{Lorenz}} Y_{k:n}$$

for $k = 1, \dots, n$. This result does finally confirm that the variability in heterogeneous exponential samples is always larger than that in homogeneous exponential samples. Consequently, a sharp lower bound for the coefficient of variation of order statistics from heterogeneous exponential random variables can be readily obtained as

$$\gamma_{X_{k:n}} \geq \sqrt{\sum_{i=1}^{k} \frac{1}{(n-i+1)^2}} \Big/ \sum_{i=1}^{k} \frac{1}{n-i+1}.$$

The rest of this paper is organized as follows. Section 2 introduces some pertinent notation and stochastic orders. The Lorenz order between order statistics from heterogeneous and homogeneous random samples is studied in Section 3, wherein the above stated general result is established. Finally, in Section 4, we mention some applications of the established result to reliability analysis and auction theory.

2. Preliminaries

In this section, we recall some stochastic orders that will be used in the sequel. Let the random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively.

Definition 2.1. X is said to be smaller than Y in the *star order*, denoted by $X \leq_\star Y$ (or $F \leq_\star G$), if the function $G^{-1}F(x)$ is star-shaped in the sense that $G^{-1}F(x)/x$ is increasing in x on the support of X.

Definition 2.2. X is said to be smaller than Y in the *convex order*, denoted by $X \leq_{\mathrm{cx}} Y$, if

$$\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$$

for all convex functions.

Definition 2.3. Let X and Y be non-negative random variables having finite positive expectations. Then, X is said to be smaller than Y in the *Lorenz order*, denoted by $X \leq_{\mathrm{Lorenz}} Y$ (or $F \leq_{\mathrm{Lorenz}} G$), if

$$\frac{X}{\mathrm{E}X} \leq_{\mathrm{cx}} \frac{Y}{\mathrm{E}Y}.$$

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