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Some prediction problems for stationary random fields with quarter-plane past

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ABSTRACT

We study several nonstandard prediction problems where a number of observations are added to the quarter-plane past of a stationary random field. The goal is to provide informative and explicit prediction error variance formulas in terms of either the autoregressive or moving average parameters of the random fields. However, unlike the time series situation, the prediction error variances for random fields seem to be expressible only in terms of the moving average parameters, and attempts to express them formally in terms of the autoregressive parameters lead to a new and mysterious projection operator which captures the nature of the "edge-effects" encountered in the estimation of the spectral density function of stationary random fields. The approach leads to a number of technical issues and open problems.

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1. Introduction

There is a fairly complete theory of stationary processes which forms the conceptual foundation for time series analysis. However, unlike the situation in time series (1-D processes), there is more than one natural choice of the past for a stationary random field with discrete time-index in the plane (2-D) or higher dimensions. Consequently, the prediction theory of stationary random fields $\{X(s,t); (s,t) \in \mathbb{Z}^2\}$ is very much underdeveloped and dependent on the choice of a past like the half-plane, cf. [11,27] and the quarter-plane, cf. [14,24–26]. The typical examples are

$$\begin{aligned} S &= \{(i,j) : i \le -1, j \in \mathbb{Z}\} \cup \{(0,j) : j \le -1\}, \\ Q &= \{(i,j) : i \le 0, j \le 0\} \setminus \{(0,0)\}, \end{aligned} \tag{1}$$

which correspond to the left half-plane and the third-quadrant, respectively. In addition, once a suitable past is chosen for a stationary random field, the associated outer factorization of its spectral density function is a more delicate analytical problem than its 1-D counterpart.

The focus of the earlier work on prediction of random fields has been on moving average (MA) representation, *one-step ahead prediction* and the extension of the Szegö–Kolmogorov–Wiener formula for the innovation variance. Our focus, however, is on using the multi-step ahead predictors and their prediction error variances to solve several nonstandard prediction problems when the third-quadrant Q is used as the past of a stationary random field and some observations are

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added to the past. More precisely, we are interested in predicting $X(0, 0)$ when a finite number of observations are added to the past Q. It turns out that solutions of such nonstandard prediction problems are not straightforward extensions of their stationary 1-D process counterparts. In fact, we are able to express the prediction error variances in terms of the MA parameters for the random field case, but attempts to express these in terms of the autoregressive (AR) parameters critically depend on a new *projection operator* which seems intrinsic to the random field situation. This projection operator accounts for the edge-effects phenomenon encountered when estimating the spectral density function of a stationary random field, cf. [3,7–9,13,17].

We assume throughout that the stationary processes $\{X(t); t \in \mathbb{Z}\}$ and $\{X(s, t); (s, t) \in \mathbb{Z}^2\}$ have zero mean and are complex-valued. For a stationary time series the nonstandard prediction problems of interest to us deal with the following modified pasts:

$$\begin{aligned} I_1 &= \{X(t); t \le -1 \text{ or } t = h\},\\ I_2 &= \{X(t); t \le h \text{ and } t \ne 0\},\\ I_3 &= \{X(t); t \le -1 \text{ and } t \ne -h\}, \end{aligned} \tag{2}$$

for any $h > 0$. The solutions corresponding to these problems are known to lead to informative and explicit expressions for the prediction error variance involving either the AR or the MA parameters, cf. [1,2,5,16,19–21,23].

Theorem 1. *Let $\{X(t); t \in \mathbb{Z}\}$ be a nondeterministic stationary process with the innovation process $\{\varepsilon(t); t \in \mathbb{Z}\}$, innovation variance σ^2, MA and AR parameters $\{b_k\}$ and $\{a_k\}$, respectively. Then, the prediction error variance of $X(0)$ based on*

(a) *the augmented past, I_1 is*

$$\operatorname{Var}\left\{X(0) - \widehat{X}_{I_1}(0)\right\} = \sigma^2 \frac{1 + |b_1|^2 + |b_2|^2 + \cdots + |b_{h-1}|^2}{1 + |b_1|^2 + |b_2|^2 + \cdots + |b_{h-1}|^2 + |b_h|^2}, \tag{3}$$

(b) *the augmented past, I_2 is*

$$\operatorname{Var}\left\{X(0) - \widehat{X}_{I_2}(0)\right\} = \sigma^2 \frac{1}{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_h|^2}, \tag{4}$$

(c) *the incomplete past, I_3 is*

$$\operatorname{Var}\left\{X(0) - \widehat{X}_{I_3}(0)\right\} = \sigma^2 \frac{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_{h-1}|^2 + |a_h|^2}{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_{h-1}|^2}. \tag{5}$$

These explicit expressions involving $\sum_{i=1}^h |b_i|^2$ and $\sum_{i=1}^h |a_i|^2$, which are reminiscent of the h-step ahead prediction error variance $\sigma^2 \sum_{i=0}^h |b_i|^2$, reveal the roles of AR and MA parameters in assessing the effect of addition or deletion of observations from the infinite past on the prediction error variance. For example, it is evident from (3), that for a stationary process with $b_h = 0$ adding the variable $X(h)$ to the past will not improve the prediction of $X(0)$. Similarly, one can see from (5), that deleting $X(-h)$ from the past will not deteriorate the prediction of $X(0)$, as long as $a_h = 0$.

The prediction error variance in (3) was obtained by [20] using spectral domain techniques and deep duality results in harmonic analysis. However, his method appeared too rigid to allow computing the predictor corresponding to the augmented past I_2, and [21] modified Nakazi's approach and developed a time-domain (regression) method to handle the other prediction problems in Theorem 1. The approach given in [21] works without requiring the unnatural minimality condition on the process and provides the linear predictor of $X(0)$ based on I_2. We study analogues of the three prediction problems in (2) for stationary random fields.

The outline of the paper is as follows: In Section 2, we review some basic results for stationary random fields, their Wold decompositions and Szegö's formula. The multi-step ahead predictor based on the quarter-plane past and its prediction error variance are obtained in terms of the MA parameters. The prediction errors based on Q for the observations in the second and fourth quadrant are represented as stationary 1-D processes, their auto-covariance and spectral density functions are also obtained. In Sections 3 and 4, we extend Theorem 1 and provide some new results for the prediction of stationary random fields corresponding to addition and deletion of observations from the past. However, the solutions are only in terms of the MA parameters, analogues of Theorem 1(b)–(c) involve some projection matrices which have proved hard to work with. These problems remain open at this time. We have focused on the third-quadrant Q, but these results can easily be modified with any other quadrant as the past. Section 5 concludes the paper.

2. Stationary random fields

Let H be the Hilbert space of zero-mean, square-integrable random variables defined on a probability space. A sequence $\{X(s, t); (s, t) \in \mathbb{Z}^2\}$ with $X(s, t) \in H$ is called a stationary random field if for all integers s_1, s_2, t_1 and t_2, the covariance of $X(s_1, t_1)$ and $X(s_2, t_2)$ depends only on the lags $(s_1 - s_2, t_1 - t_2)$, namely,

$$\operatorname{Cov}(X(s_1, t_1), X(s_2, t_2)) = \gamma(s_1 - s_2, t_1 - t_2).$$