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On usual multivariate stochastic ordering of order statistics from heterogeneous beta variables

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ABSTRACT

Let $X_i \sim \text{beta}(\alpha_i, 1)$ and $Y_i \sim \text{beta}(\gamma_i, 1)$, $i = 1, 2$, be all independent. We show that $(\alpha_1, \alpha_2) \overset{m}{\succeq} (\gamma_1, \gamma_2)$ implies $(Y_{1:2}, Y_{2:2}) \geq_{st} (X_{1:2}, X_{2:2})$. We then extend this result to the general case of the proportional reversed hazard rates (PRHR) model.

1. Introduction

Order statistics have been studied quite extensively in the literature due to the key role they play in many areas of statistics. They especially play a special role in reliability theory wherein they correspond to k-out-of-n systems. Such a system, consisting of n components, works as long as at least k components work. Let $X_1, \ldots, X_n$ denote the lifetimes of the components and $X_{1:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Then, $X_{n-k+1:n}$ corresponds to the lifetime of such a k-out-of-n system. So, many properties of k-out-of-n systems have been established in the literature by using the theory of order statistics; see [1] for some recent results in this direction. Interested readers may refer to Balakrishnan and Rao [2,3] for elaborate discussions on theory and applications of order statistics.

Let $X_1, \ldots, X_n$ denote the lifetimes of n components of a system with distribution functions $F_1, \ldots, F_n$, respectively. Then, $X_1, \ldots, X_n$ are said to follow the PRHR model if there exist positive constants $\alpha_1, \ldots, \alpha_n$ and a distribution function $F(x)$ with corresponding density function $f(x)$ such that $F_i(x) = F^{\alpha_i}(x)$ for $i = 1, \ldots, n$. In this case, $F(x)$ and $\tilde{r}(x) = f(x)/F(x)$ are called the baseline distribution and baseline reversed hazard functions, respectively, and $\alpha_1, \ldots, \alpha_n$ are the proportional reversed hazard rate parameters. Distributions such as power, generalized exponential and exponentiated Weibull are all special cases of this model. One may refer to Chapter 7 of Marshall and Olkin [5] for a discussion on this model.

We now review briefly some common notions of stochastic orders and majorization order. Throughout, the terms *increasing* and *decreasing* are used for *non-decreasing* and *non-increasing*, respectively. Let X and Y be two random variables with distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, and density functions f and g, respectively.

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X is said to be larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$) if $\bar{F}(x) \geq \bar{G}(x)$. A multivariate version of the usual stochastic order is as follows. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two random vectors. Then, $\mathbf{X}$ is said to be larger than $\mathbf{Y}$ in the usual multivariate stochastic order (denoted by $\mathbf{X} \geq_{st} \mathbf{Y}$) if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all increasing functions $\phi : \mathbb{R}^n \to \mathbb{R}$. It is easy to see that the multivariate stochastic ordering implies componentwise usual stochastic ordering. Interested readers may refer to Müller and Stoyan [9] and Shaked and Shanthikumar [11] for detailed discussions on univariate and multivariate stochastic orderings.

For two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, let $\{x_{(1)}, \ldots, x_{(n)}\}$ and $\{y_{(1)}, \ldots, y_{(n)}\}$ denote the increasing arrangements of their components, respectively. A vector $\boldsymbol{x}$ is said to majorize another vector $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{m}{\geq} \boldsymbol{y}$) if $\sum_{j=1}^{i} x_{(j)} \leq \sum_{j=1}^{i} y_{(j)}$ for $i = 1, \ldots, n-1$, and $\sum_{j=1}^{n} x_{(j)} = \sum_{j=1}^{n} y_{(j)}$. A real-valued function ϕ defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathbb{A}$ if $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$ implies $\phi(\boldsymbol{x}) \geq (\leq) \phi(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}$. One may refer to Marshall et al. [6] for a detailed discussion on majorization and Schur-type functions.

In this work, we obtain some new results about stochastic comparison of vectors of order statistics. Specifically, taking $X_i \sim \text{beta}(\alpha_i, 1)$ and $Y_i \sim \text{beta}(\gamma_i, 1)$, $i = 1, 2$, all being independent, we prove that

$$(\alpha_1, \alpha_2) \overset{m}{\succeq} (\gamma_1, \gamma_2) \Longrightarrow \Big(Y_{1:2}, Y_{2:2}\Big) \geq_{st} \Big(X_{1:2}, X_{2:2}\Big). \tag{1.1}$$

We further extend the result in (1.1) to the general PRHR model. It is useful to mention here that the beta distributions considered here are also special cases of the Kumaraswamy distributions [4].

2. Main results

In this section, we consider stochastic comparison of vectors of order statistics in the PRHR model. In this case, the beta distribution mentioned above is the simplest element of the PRHR model. Also, some other known distributions belonging to the PRHR model can be derived by a simple transform on the beta distribution. For this reason, we first focus on this case and present a result concerning some properties of order statistics arising from heterogeneous beta random variables. The following theorem gives necessary and sufficient conditions for characterizing Schur-convex and Schur-concave functions.

Theorem 2.1 *([6, p. 84]). Let $I \subset \mathbb{R}$ be an open interval and $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Then, the necessary and sufficient conditions for ϕ to be Schur-convex on I^n are ϕ is symmetric on I^n and for all $i \neq j$,*

$$(z_i - z_j)\left(\frac{\partial \phi}{\partial z_i}(\mathbf{z}) - \frac{\partial \phi}{\partial z_j}(\mathbf{z})\right) \geq 0 \quad \text{for all } \mathbf{z} \in I^n,$$

where $\frac{\partial \phi}{\partial z_i}(\mathbf{z})$ denotes the partial derivative of ϕ with respect to its ith argument. Function ϕ is Schur-concave if and only if it is symmetric and the reversed inequality sign holds in the above inequality.

Lemma 2.1. *Let $X_i \sim \text{beta}(\alpha_i, 1)$, $i = 1, 2$, be independent. Then,*

(i) $X_{2:2} \sim \text{beta}(\alpha_1 + \alpha_2, 1)$;
(ii) *$X_{1:2}/X_{2:2}$ and $X_{2:2}$ are independent;*
(iii) *the distribution function of $X_{1:2}/X_{2:2}$ is Schur-convex in (α_1, α_2).*

Proof. (i) The distribution function of $X_{2:2}$, for $x \in (0, 1)$, is

$$F_{X_{2:2}}(x) = F_{X_1}(x) F_{X_2}(x) = x^{\alpha_1+\alpha_2}, \quad x \in (0, 1),$$

and so Part (i) is immediate.

(ii) The joint density function of $(X_{1:2}, X_{2:2})$ is given by

$$\begin{aligned} f(x_1, x_2) &= f_{X_1}(x_1) f_{X_2}(x_2) + f_{X_1}(x_2) f_{X_2}(x_1) \\ &= \alpha_1\alpha_2 \Big(x_1^{\alpha_1-1} x_2^{\alpha_2-1} + x_1^{\alpha_2-1} x_2^{\alpha_1-1}\Big) I_{(0,1)}(x_1)\, I_{(x_1,1)}(x_2), \end{aligned}$$

where $I_A(x) = 1$ if and only if $x \in A$. Let $U_1 = X_{1:2}/X_{2:2}$ and $U_2 = X_{2:2}$. Now, to prove the required result, we must show that

$$f_{U_1,U_2}(x_1, x_2) = f_{U_1}(x_2) f_{U_2}(x_2) \quad \text{for all } x_1, x_2, \tag{2.2}$$

where $f_{U_1,U_2}(.,.)$ is the joint density function of (U_1, U_2), and $f_{U_i}(.)$ are the density functions of U_i, $i = 1, 2$. It is easy to see that

$$\begin{aligned} f_{U_1,U_2}(x_1, x_2) &= x_2 f(x_1x_2, x_2) \\ &= \left\{\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\Big(x_1^{\alpha_1-1} + x_1^{\alpha_2-1}\Big) I_{(0,1)}(x_1)\right\}\left\{(\alpha_1+\alpha_2)\, x_2^{\alpha_1+\alpha_2-1}\, I_{(0,1)}(x_2)\right\} \\ &= f_{U_1}(x_1) f_{U_2}(x_2), \end{aligned}$$

where the last equality follows from Part (i). Thus, Part (ii) follows.