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On construction of general classes of bivariate distributions

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ABSTRACT

In this study, taking into account the physics of failure (death) and the interrelationship between the items involved, we propose a general methodology for constructing new 'classes of bivariate distributions'. The approach is based on the stochastic modeling of the residual lifetime of an item after the failure of the other item. We derive the joint and the corresponding marginal distributions in a class constructed from the modeling of conditional failure rate. It is shown that the proposed new class includes several well-known bivariate distributions as special bivariate models. As illustrated, a number of new families of bivariate distributions are generated from the new class proposed in this paper. Furthermore, we briefly discuss the relationship to Freund's bivariate exponential distribution.

1. Introduction

Bivariate distributions are very important in modeling dependent random quantities in many different areas such as reliability, survival analysis, queueing analysis, insurance risk analysis, life insurance, and so forth (see, e.g., Iyer and Manjunath [20]). Especially, in the area of the lifetime modeling and analysis, the lifetimes of organisms or items are most often stochastically dependent. For an example, it is observed that there is a very high positive correlation between the times of deaths of coupled lives (see Carriere [6]). It is also addressed that, after the marital bereavement, the risk of mortality is significantly increased (see Jagger and Sutton [21]). In this regard, there have been numerous bivariate models proposed in the literature.

Traditionally, during the initial period of study on the topic, many researchers tried to extend the exponential distribution to the bivariate case. For instance, the work by Gumbel [16] can be considered as the initial work in this direction. Then, Freund [13] proposed a bivariate extension of the exponential distribution, which is absolutely continuous. Marshall and Olkin [27] also proposed a very important bivariate exponential distribution but it is not absolutely continuous. Thus, as illustrated in Section 9 of Block and Basu [5], there are clearly situations when the model of Marshall and Olkin [27] cannot be applied. Further extensions and discussions were also performed by, e.g., Downton [11], Hawkes [17], Block and Basu [5], Shaked [31], Sarkar [30] and Hayakawa [18]. Motivated by the work of Freund [13], Spurrier and Weier [33] derived a bivariate survival model based on the Weibull distribution. In the literature, extensions of other distributions to the bivariate models have also been proposed (see e.g., Lee [26], Sarhan and Balakrishnan [29]). A review on several theoretical bivariate gamma distributions with gamma marginals can be found in Yue et al. [34]. A nice review on the modeling of multivariate survival models can also be found in Hougaard [19]. An excellent encyclopaedic survey for various bivariate distributions can be found in Balakrishnan and Lai [4].

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Until now, in addition to the aforementioned works, numerous models for modeling general bivariate distributions have been proposed. One of the most convenient ways of constructing a bivariate distribution is that based on a copula function (see Nelson [28]). However, in the approach based on copula functions, the models are frequently lacking in the physics of dependency which lies behind the models. Also in the approaches other than the copula function, very few have studied whether the derivation of the bivariate models actually fits the way how the stochastic dependency is supposed to be generated.

On the contrary, in this paper, taking into account the physics of failure (death) of items (organisms) and the interrelationship between them, we propose and discuss a general methodology for constructing new 'general classes of bivariate distributions'. Our approach is based on the stochastic modeling of the residual lifetime of the remaining item after the failure of the other item. It is worthy of note again that this paper proposes not a specific parametric family of distributions, but general methodology for constructing 'classes of parametric families' of distributions. It is also shown that the proposed classes allow much flexibility in modeling bivariate distributions. Although our study had started independent of the well-known bivariate distributions such as Freund [13] and Block and Basu [5], it is shown that the proposed new class includes these several well-known bivariate distributions as special bivariate distributions belonging to the class. One of the most important contributions of this paper is that it provides a new 'general insight' and 'new perspective' on the modeling of the bivariate distributions.

The construction of this paper is as follows. In Section 2, a general methodology for constructing new classes of bivariate distributions is suggested and a new class is constructed. We start our main discussion with the modeling based on the concept of failure rate process and the failure rate order relationship. It will be briefly explained how other stochastic orderings applied to the modeling of residual lifetimes can generate other classes of distributions. In Section 3, a number of new families of bivariate distributions are generated from the new class proposed in Section 2. Especially, the relationship to Freund's bivariate exponential distribution will be discussed in detail. Finally, in Section 4, the results in this paper are briefly summarized. Furthermore, some topics for the future study are suggested and concluding remarks are given.

2. Construction of general class of bivariate distributions

In this section, we will construct a new class of bivariate distributions considering the physics of failure (death) of items (organisms) and the interrelationship between them. For a convenient description of the approach and procedure, we will discuss the model in terms of reliability context such as 'component' and 'failures'. However, the application of the proposed model is not necessarily limited to the area of reliability, but it can generally be applied to the modeling of dependent lifetimes in survival analysis, life insurance, biomedical areas, and so on.

Consider a system composed of two components. In many practical situations, when the failure of one component occurs in such a system, the remaining component may suffer from more stress or increased load. This could significantly affect the residual lifetime of the remaining component, eventually shortening the residual lifetime of the remaining component. This kind of dependency can frequently be observed in different practical situations, for example, in the survival of the paired organs such as person's eyes, ears, kidneys and lungs, or in the survival of two-engine airplane, or in the failures of adjacent pumps on a cooling system.

We will now start to discuss our model in a more detail. Suppose that the system is composed of two components: component 1 and component 2 and they start to operate at time $t = 0$. The original lifetimes of components 1 and 2, when they start to operate, are described by the corresponding failure rates $\lambda_1(t)$ and $\lambda_2(t)$, respectively. These original lifetimes of components 1 and 2 are denoted by X_1^* and X_2^*, respectively, assuming that X_1^* and X_2^* are stochastically independent.

We consider the practical situation when the failure of one component increases the stress of the other component, which results in the shortened residual lifetime of the remaining component. Thus, there is a change point, $\min\{X_1^*, X_2^*\}$, after which the residual lifetime distribution of the remaining component changes. Under this type of dependency, we denote the corresponding eventual lifetimes of components 1 and 2 by X_1 and X_2, respectively. In order to describe the stochastic dependence model more precisely, 'for a moment' in the next paragraph, we need to discuss the concept of 'conditional failure rate' under a general setting, which is crucial for a proper understanding of our model to be discussed.

Assume that a device or an organism is operating (living) in a random environment described by a certain (covariate) stochastic process $\{Z(t), t \geq 0\}$. The lifetime of this device or organism is denoted by T. For example, the stochastic process $\{Z(t), t \geq 0\}$ can represent the randomly changing time-dependent external temperature, electric or mechanical load, or some other randomly changing external stress, etc. Then, the *conditional failure rate* can formally be defined (see Kalbfleisch and Prentice [22]):

$$r(t|z(s), 0 \leq s \leq t) \equiv \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | Z(s) = z(s), 0 \leq s \leq t, T > t)}{\Delta t}.$$

Note that this conditional failure rate can be specified for a realization of the covariate process. With the covariate process not fixed yet, it is also obviously a stochastic process, which is usually referred to as the 'failure (hazard) rate process' (or random failure rate). For details, see, e.g., Kebir [23], Aven and Jensen [2] and Finkelstein and Cha [12]. Then, based on it, the conditional survival function can now be specified as

$$P(T > t | Z(s) = z(s), 0 \leq s \leq t) = \exp\left\{-\int_0^t r(w|z(s), 0 \leq s \leq w) dw\right\}. \tag{1}$$