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On modular decompositions of system signatures

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ABSTRACT

Considering a semicoherent system made up of n components having i.i.d. continuous lifetimes, Samaniego defined its structural signature as the n-tuple whose kth coordinate is the probability that the kth component failure causes the system to fail. This n-tuple, which depends only on the structure of the system and not on the distribution of the component lifetimes, is a very useful tool in the theoretical analysis of coherent systems.

It was shown in two independent recent papers how the structural signature of a system partitioned into two disjoint modules can be computed from the signatures of these modules. In this work we consider the general case of a system partitioned into an arbitrary number of disjoint modules organized in an arbitrary way and we provide a general formula for the signature of the system in terms of the signatures of the modules.

The concept of signature was recently extended to the general case of semicoherent systems whose components may have dependent lifetimes. The same definition for the n-tuple gives rise to the probability signature, which may depend on both the structure of the system and the probability distribution of the component lifetimes. In this general setting, we show how under a natural condition on the distribution of the lifetimes, the probability signature of the system can be expressed in terms of the probability signatures of the modules. We finally discuss a few situations where this condition holds in the non-i.i.d. and nonexchangeable cases and provide some applications of the main results.

1. Introduction

We consider an n-component system $S = (C, \phi, F)$, where C is the set $[n] = \{1, \ldots, n\}$ of components, $\phi\colon \{0, 1\}^n \to \{0, 1\}$ is the structure function (which expresses the state of the system in terms of the states of its components), and F denotes the joint c.d.f. of the component lifetimes $T_1, \ldots, T_n$, that is,

$$F(t_1, \ldots, t_n) = \Pr(T_1 \leqslant t_1, \ldots, T_n \leqslant t_n), \quad t_1, \ldots, t_n \geqslant 0.$$

We assume that the system is *semicoherent*, i.e., the structure function ϕ is nondecreasing[1] in each variable and satisfies the conditions $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$. We also assume that the c.d.f. F has no ties, that is, $\Pr(T_i = T_j) = 0$ for all distinct $i, j \in [n]$.

The concept of *signature* was introduced in 1985 by Samaniego [12], for systems whose components have continuous and i.i.d. lifetimes, as the n-tuple $\mathbf{s} = (s_1, \ldots, s_n)$ whose kth coordinate s_k is the probability that the kth component failure

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1 Such a function is also known as a monotone function.

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causes the system to fail. In other words, we have

$$s_k = \Pr(T_S = T_{k:n}), \quad k \in [n],$$

where T_S denotes the system lifetime and $T_{k:n}$ denotes the kth smallest lifetime, i.e., the kth order statistic obtained by rearranging the variables $T_1, \ldots, T_n$ in ascending order of magnitude.

It was shown in [2] that s_k can be explicitly written in the form[2]

$$s_k = \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} \frac{1}{\binom{n}{|A|}} \phi(A) - \sum_{\substack{A \subseteq C \\ |A|=n-k}} \frac{1}{\binom{n}{|A|}} \phi(A). \tag{1}$$

This formula shows that, in the i.i.d. case, the probability $\Pr(T_S = T_{k:n})$ does not depend on the distribution F of the component lifetimes. Thus, the system signature is a purely combinatorial object associated with the structure ϕ. Due to this feature, in both the i.i.d. and non-i.i.d. cases the n-tuple $\mathbf{s} = (s_1, \ldots, s_n)$, where s_k is defined by (1), is referred to as the *structural signature* of the system.

Since its introduction the concept of structural signature proved to be a very useful tool in the analysis of semicoherent systems, especially for the comparison of different system designs and the computation of the system reliability (see [13]).

The interest of extending the concept of signature to the general case of dependent lifetimes has been pointed out in several recent papers. Just as in the i.i.d. case, we can consider the n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$, called *probability signature*, whose kth coordinate is the probability $p_k = \Pr(T_S = T_{k:n})$. Thus defined, the probability signature obviously coincides with the structural signature when the component lifetimes are i.i.d. and continuous. Actually, it is easy to see that both concepts also coincide when the lifetimes are exchangeable and the distribution F has no ties; see, e.g., [8,9] for more details. However, these two concepts are generally different. Contrary to the structural signature, the probability signature may depend on the distribution of the component lifetimes. It is then considered as a probabilistic object associated with both the structure ϕ and the distribution F; see [6,7,14,10] for basic properties of this concept.

Even in the i.i.d. (or exchangeable) case, the computation of the signature may be a hard task when the system has a large number of components. However, the computation effort can be greatly reduced when the system is decomposed into distinct modules (subsystems) whose structural signatures are already known.

First results along this line were presented in [3,4]. In particular, in [4] explicit expressions for the structural signatures of systems consisting of two modules connected in series or in parallel were provided in terms of the structural signatures of the modules. A general procedure to compute the structural signatures of recurrent systems (i.e., systems partitioned into identical modules) was also described. Moreover, the key role of the concepts of tail and cumulative signatures were pointed out (see definitions in Section 2).

In this work we extend these results in the following two directions:

1. Considering the general case of a system partitioned into an arbitrary number of disjoint modules connected according to an arbitrary semicoherent structure, we yield an explicit formula for the *modular decomposition of the structural signature of the system*, that is, an explicit expression of the structural signature of the system only in terms of the structural signatures of the modules and the structure of the modular decomposition (i.e., the structure that defines the way the modules are interconnected). This result, which holds without any additional assumption and is obviously independent of the distribution F of the component lifetimes, is presented in Section 2.
2. Considering again the general case of systems partitioned into an arbitrary number of disjoint modules, we show that a similar modular decomposition of the probability signature still holds if and only if the distribution of the component lifetimes (i.e., the function F) satisfies a natural decomposition condition (associated with the decomposition of the system into modules). Thus, a modular decomposition of the probability signature appears whenever two decomposition properties hold: a structural decomposition of the system into modules combined with a decomposition of the distribution of the component lifetimes. We also yield an explicit formula for this modular decomposition of the probability signature. This result is presented in Section 3. Also, we note that the proofs of our decomposition formulas are simpler than those in [3,4].

It is noteworthy that both the structural and probability signatures of the system can be computed by our modular decomposition formulas without knowing the structures of the modules. Only the knowledge of the signatures of the modules and the structure of the modular decomposition (i.e., the way the modules are connected) is required. Thus, the computation of the signature of a large system can be made much easier when it is decomposed into a small number of modules whose signatures are known.

In Section 4 we discuss and demonstrate our results through a few examples and provide an interpretation of the new concept of decomposition of the distribution. Some concluding remarks are then given in Section 5.

[2] As usual, we identify Boolean vectors $\mathbf{x} \in \{0, 1\}^n$ and subsets $A \subseteq [n]$ by setting $x_i = 1$ if and only if $i \in A$. We thus use the same symbol to denote both a function $f: \{0, 1\}^n \to \mathbb{R}$ and the corresponding set function $f: 2^{[n]} \to \mathbb{R}$, interchangeably. For instance, we write $\phi(0, \ldots, 0) = \phi(\varnothing)$.