Journal of Multivariate Analysis 129 (2014) 171–185

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

On the estimation of the medial axis and inner parallel body

Antonio Cuevas [a], Pamela Llop [b,c], Beatriz Pateiro-López [d,*]

[a] Departamento de Matemáticas, Universidad Autónoma de Madrid, Spain

[b] Facultad de Ingeniería Química (UNL), Argentina

[c] Instituto de Matemática Aplicada del Litoral (UNL - CONICET), Argentina

[d] Departamento de Estadística e Investigación Operativa, Universidad de Santiago de Compostela, Spain

ARTICLE INFO

Article history:
Received 23 June 2013
Available online 30 April 2014

AMS subject classifications:
62G05
62G20

Keywords:
Image analysis
Set estimation
r-convexity
Skeleton
Devroye–Wise estimator

ABSTRACT

The medial axis and the inner parallel body of a set C are different formal translations for the notions of the "central core" and the "bulk", respectively, of C. On the basis of their applications in image analysis, both notions (and especially the first one) have been extensively studied in the literature, from different points of view. A modified version of the medial axis, called λ-medial axis, has been recently proposed in order to get better stability properties. The estimation of these relevant subsets from a random sample of points is a partially open problem which has been considered only very recently. Our aim is to show that standard, relatively simple, techniques of set estimation can provide natural, consistent, easy-to-implement estimators for both the λ-medial axis $\mathcal{M}_\lambda(C)$ and the inner parallel body $I_\lambda(C)$ of C. The consistency of these estimators follows from two results of stability (i.e. continuity in the Hausdorff metric) of $\mathcal{M}_\lambda(C)$ and $I_\lambda(C)$ obtained under some, not too restrictive, regularity assumptions on C. As a consequence, natural algorithms for the approximation of the λ-medial axis and the λ-inner parallel body can be derived. The whole approach could be useful for some practical problems in image analysis where estimation techniques are needed.

1. Introduction

General statement of the problem. The plan of this work.

There is a rich mathematical literature devoted to the study of the "central part" of a set C in the Euclidean space (which would represent, in statistical terms, the "median of C"); see [4]. Of course, the first step in any such study must be to give a precise meaning to this loose notion of "set median". Different definitions, closely related but not always equivalent, have been proposed. The most popular one is perhaps the *medial axis* of C, $\mathcal{M}(C)$, defined as the subset of points in C having at least two projections on the boundary ∂C. Other closely related (but not equivalent) usual notions are the *skeleton*, $\mathcal{S}(C)$, (the set of centers of maximal balls included in C) and the *cut locus* of C, defined as the topological closure of $\mathcal{M}(C)$; see below for further discussion on these notions. The medial axis was introduced by Blum [8] as a tool in image analysis. The papers by Chazal and Soufflet [12], Chazal and Lieutier [11] and Chaussard et al. [10], among many others, analyze these ideas from different points of view.

We are especially concerned with a modified version of the medial axis, called λ-*medial axis*, $\mathcal{M}_\lambda(C)$, introduced in [11] to deal with the well-known problem of instability in the medial axis: the medial axis $\mathcal{M}(C)$ and $\mathcal{M}(D)$ might be far away from

* Correspondence to: Facultad de Matemáticas, Rúa Lope Gómez de Marzoa s/n, 15782 Santiago de Compostela, Spain.
E-mail addresses: antonio.cuevas@uam.es (A. Cuevas), pllop@santafe-conicet.gov.ar (P. Llop), beatriz.pateiro@usc.es (B. Pateiro-López).

http://dx.doi.org/10.1016/j.jmva.2014.04.011

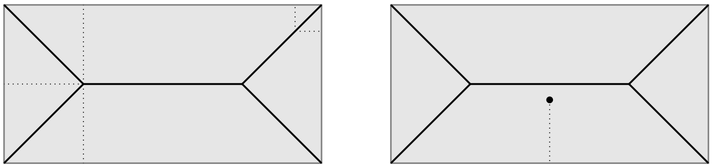

Fig. 1. The left panel shows the medial axis of a rectangle. The right panel shows a point which does not belong to the medial axis since it has only one closest boundary point.

each other even if the original sets C and D and their boundaries are very close together; see [12] and references therein. The λ-medial axis leaves out those points of $\mathcal{M}(C)$ whose metric projections on ∂C are too close together.

Another, perhaps less popular, closely related concept is the so-called λ-*inner parallel body*, $I_\lambda(C)$, defined as the set of points in C whose distance to ∂C is at least λ. So far this concept has been mainly studied in the case where C is convex [see, e.g., [36]] but we will see that this assumption is not necessary to find a simple consistent estimator of $I_\lambda(C)$. The λ-inner parallel body has a simple intuitive interpretation and is obviously close to the notion of "core" of C. In some cases it provides an outer approximation to the λ-medial axis. The algorithm proposed in the paper by Genovese et al. [26] for medial axis estimation (under a regression-type sampling model) relies on an estimate of the inner parallel set.

This paper deals with the statistical problem of estimating the λ-medial axis, $\mathcal{M}_\lambda(C)$, and the λ-inner parallel body, $I_\lambda(C)$, from a random sample of points $X_1, \ldots, X_n$ drawn inside C. The whole approach relies on a simple plug-in idea: we will use methods of set estimation (see, e.g., [15] for a survey) to get a suitable estimator $C_n = C_n(X_1, \ldots, X_n)$ of C. Then the natural estimators of $\mathcal{M}_\lambda(C)$ and $I_\lambda(C)$ would be just $\mathcal{M}_\lambda(C_n)$ and $I_\lambda(C_n)$, respectively.

Whereas the theoretical and practical aspects of the medial axis (and associated notions) have received a considerable attention, the problem of estimating this set has been considered only very recently: we refer to the recent paper by Genovese et al. [26] for a general approach to the estimation of $\mathcal{M}(C)$ and $I_\lambda(C)$, though the sampling model considered by these authors is a bit different to that we will use here. In short, we will show that imposing an additional shape restriction on C (called r-*convexity*) one can obtain, in return, a considerable simplification in the theory and practice of the estimation of $\mathcal{M}_\lambda(C)$ and $I_\lambda(C)$.

Some basic definitions.

The original idea of medial axis was proposed by Blum [8] as a tool in image analysis. Let C be a bounded set in $\mathbb{R}^d$ with non-empty interior such that C is *regular*, i.e., $C = \overline{\text{int}(C)}$. Following the notation in [30], for any $x \in C$, let us denote by $\Gamma(x)$ the set of boundary points closest to x, that is,

$$\Gamma(x) = \{y \in \partial C : d(x, y) = d(x, \partial C)\},$$

where $d(x, y) = \|x - y\|$ denotes the Euclidean distance between x and y in $\mathbb{R}^d$ and also (with a slight notational abuse) for a set $A \subset \mathbb{R}^d$, $d(x, A) = \inf\{d(x, y) : y \in A\}$.

We can now establish the main definition:

The ***medial axis*** *of* C, $\mathcal{M}(C)$, *is the set of points* x *of* C, *that have at least two closest boundary points, that is,* $\mathcal{M}(C) = \{x \in C : |\Gamma(x)| \geq 2\}$ *where* $|E|$ *denotes the cardinal of* E.

It is easy to see that $\mathcal{M}(C) \neq \emptyset$. The medial axis of a rectangle is shown in Fig. 1. We will follow here the above definition although other authors (in particular, in the theory of surface reconstruction) use a slightly different notion: if Σ is a curve or a surface, the medial axis of Σ is sometimes defined, see e.g. [20], as the closure of the set of points (in the whole space) having at least two closest points in Σ.

The medial axis, $\mathcal{M}(C)$, combined with the radius function (the distance from every point in $\mathcal{M}(C)$ to the closest boundary point), provides the so-called *medial axis transform*, a powerful tool in computer vision; see [14] for mathematical aspects of this notion and further references.

Remark 1. There is no universal agreement on the precise meaning of the term "medial axis". We have followed here what seems to be the consensus in the recent literature. However, in addition to the option of considering or not the external points of C in the medial axis, there are two other closely related formal versions of the same idea, called "skeleton" and "cut locus", which appear sometimes confounded with the medial axis. Just for the sake of clarity and completeness, we next introduce and briefly comment these concepts.

The *skeleton* of C, $\mathcal{S}(C)$, is the set of centers of maximal balls inscribed in the set. This is, the set of centers of those balls included in the set C that are not contained in any another ball included in C.

The *cut locus* of C is the topological closure of the medial axis of C. It can be proved that $\mathcal{M}(C) \subseteq \mathcal{S}(C) \subseteq \overline{\mathcal{M}(C)}$.

An example of a set C for which the skeleton is not closed, and the last inclusion is strict, can be found in [12]. Some further interesting analytic and topological properties are obtained in [24]. Note that the notion of central set and skeleton used by this author coincide, respectively, with the concept of skeleton and medial axis defined above.

As mentioned above, a major problem in the practical use of the medial axis $\mathcal{M}(C)$ lies in its instability with respect to slight perturbations in the boundary ∂C, see Fig. 2 (left). A natural idea in order to "robustify" $\mathcal{M}(C)$ has been proposed by