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Geometric interpretation of the residual dependence coefficient

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ABSTRACT

The residual dependence coefficient was originally introduced by Ledford and Tawn (1996) [25] as a measure of residual dependence between extreme values in the presence of asymptotic independence. We present a geometric interpretation of this coefficient with the additional assumptions that the random samples from a given distribution can be scaled to converge onto a limit set and that the marginal distributions have Weibull-type tails. This result leads to simple and intuitive computations of the residual dependence coefficient for a variety of distributions.

1. Introduction

From an applied perspective, model selection often requires some knowledge of dependence between the components of a multivariate random vector. In particular, when the interest is in risk assessment and risk management, dependence in extreme values is of primary importance. If a portfolio comprises several risky assets, while losses on one or some of the assets have the potential to lead to a substantial loss on the overall portfolio, the situation is exacerbated in the presence of positive dependence at extreme and sub-extreme loss levels.

Several measures have been proposed in the literature to quantify dependence in extremes. We restrict attention to the upper tails of the marginal distributions. Let (X, Y) be a bivariate random vector with continuous marginal distribution functions (dfs) F_1 and F_2, respectively. The *tail dependence coefficient* of X and Y is defined as

$$\lambda := \lim_{t\to\infty} \frac{\Pr\{F_1(X) > 1 - 1/t, F_2(Y) > 1 - 1/t\}}{\Pr\{F_1(X) > 1 - 1/t\}}, \tag{1.1}$$

provided the limit exists; see e.g. [23,29]. When $\lambda > 0$, X and Y are said to be *asymptotically dependent*; whereas when $\lambda = 0$, X and Y are said to be *asymptotically independent*. The terminology is best illustrated in the context of extreme value theory. Suppose $\lambda = 0$. Under the additional assumption that dfs F_1 and F_2 belong to the maximum domain of attraction of an extreme value limit law, the partial maxima of random samples from F_1 and F_2, under suitable affine normalizations, converge in distribution as the sample size tends to infinity to a product measure, and hence are asymptotically independent. The criterion $\lambda = 0$ for asymptotic independence is due to [35].

In [5], the authors propose an alternative criterion for asymptotic independence, which is based on the asymptotic shape of the scaled random samples. A bivariate set D is said to be *blunt* if the coordinate-wise supremum of the points in D does not

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lie in the closure of D. Under certain regularity conditions on D, if the random samples from the joint distribution of random vector (X, Y) can be scaled to converge on the limit set D and D is blunt, then X and Y are asymptotically independent (Theorem 4.1 in [5]).

The condition $\lambda = 0$ for asymptotic independence means that the probability of X and Y simultaneously exceeding high quantiles decays to zero faster than the marginal probability of exceedance as the quantile level approaches one. The tail dependence coefficient λ does not capture the speed of convergence of the ratio of probabilities in (1.1) to zero. Similarly, the condition of bluntness of the limit set does not take into account the precise shape of the limit set, at least in the upper right corner of the enclosing rectangle.

A classic example of a model with asymptotic independence is the normal distribution; see e.g. [35]. In this case, when the margins are standard normal and have correlation $\rho \in (-1, 1)$,

$$\Pr\{F_1(X) > 1 - 1/t,\ F_2(Y) > 1 - 1/t\} \sim c_\rho\, t^{-2/(1+\rho)}\, (\log t)^{-\rho/(1+\rho)}, \quad t \to \infty,$$

where $c_\rho = (4\pi)^{-r/(1+\rho)}(1+\rho)^{3/2}(1-\rho)^{-1/2}$; see [31,25]. The result shows the impact of the correlation on the rate of decay of the joint exceedance probability. Level sets of a bivariate normal density are all scaled copies of an ellipse determined by the covariance matrix, and so is the limit set for the normal random samples; see [17]. For $\rho = 0$, the limit set is the unit ball in the Euclidean norm; as $\rho > 0$ increases the elliptic limit set becomes more and more elongated along the main diagonal, thus exemplifying the influence of ρ on its shape.

In order to capture the residual dependence between extremes in the presence of asymptotic independence as defined above, a new measure was introduced in [25]; see also [26,27,22]. Statistical estimation and applications of this measure are treated, for example, in [10,30,14]. As before, let (X, Y) be a bivariate random vector with continuous marginal dfs F_1 and F_2. Suppose, for $x, y > 0$, the limit

$$\lim_{t\to\infty} \frac{\Pr\{F_1(X) > 1 - x/t,\ F_2(Y) > 1 - y/t\}}{\Pr\{F_1(X) > 1 - 1/t,\ F_2(Y) > 1 - 1/t\}}$$

exists and is positive. This implies that the joint probability of exceedance

$$\Pr\{F_1(X) > 1 - 1/t,\ F_2(Y) > 1 - 1/t\}$$

varies regularly at infinity with index $-1/\eta$ for some $\eta \in (0, 1]$; see [32], Section 5.4.2 and [12], Section 7.6. It follows from Proposition 0.8(i) in [32] that the constant η is also the limit of the ratio of log-probabilities:

$$\lim_{t\to\infty} \frac{\log \Pr\{F_1(X) > 1 - 1/t\}}{\log \Pr\{F_1(X) > 1 - 1/t, F_2(Y) > 1 - 1/t\}} = \eta, \tag{1.2}$$

using the fact that $\log \Pr\{F_1(X) > 1 - 1/t\} = -\log t$. As the probability in the denominator of (1.2) is too small, a limit is achieved by considering log-probabilities. This is in the spirit of large deviations theory. The value $\eta < 1$ implies asymptotic independence. The constant η is referred to as the *residual dependence coefficient* to emphasize the fact that it measures the degree of residual dependence when marginal extremes are asymptotically independent.

The direct analytic computations of the residual dependence coefficient based on its definition can be fairly involved. There exist a number of papers that focus on the sole computation of this coefficient for specific families of distributions. In the original paper Ledford and Tawn [25], the residual dependence coefficient is computed for the bivariate normal distribution; Heffernan [21] presents results for several bivariate distributions in the copula representation; Hashorva [18] considers elliptically symmetric distributions with an extension to scale mixture distributions given in [19]; and recently Schlueter and Fischer [34] computed the coefficient η for the elliptic (symmetric) generalized hyperbolic distribution. Results of the latter paper have been further generalized in [20]. Both elliptical and generalized hyperbolic distributions play an important role in financial applications; see [29] and references therein.

In the main result of the paper, Theorem 2.1, we show that the residual dependence coefficient has a simple geometric interpretation in terms of the shape of the limit set. This interpretation facilitates analytic computations of η. It also reveals a new intuitive meaning behind the concept of the residual dependence coefficient. Our approach allows us to handle the special cases mentioned above as well as their extensions in a unified way; this is illustrated in Section 3 by examples. Finally, Section 4 investigates the impact of linear transformations of the underlying random vectors on the residual dependence coefficient.

2. Residual dependence coefficient and the limit set

As mentioned in the introduction, we would like to investigate the link between the asymptotic shape of the scaled random samples and the asymptotic dependence properties of the underlying distribution. We first make precise the notion of the asymptotic shape of random samples. Consider a sequence of independent and identically distributed random vectors $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ on $\mathbb{R}^d$. Let $N_n := \{\mathbf{Z}_1/r_n, \ldots, \mathbf{Z}_n/r_n\}$ denote an n-point *sample cloud* with scaling constants $r_n > 0$, $r_n \to \infty$ as $n \to \infty$. We use notation $N_n(A) = \sum_{i=1}^n \mathbf{1}_A(\mathbf{Z}_i/r_n)$ for a Borel set $A \subset \mathbb{R}^d$, where $\mathbf{1}_A(\cdot)$ denotes the indicator function with $\mathbf{1}_A(\mathbf{x}) = 1$ if $\mathbf{x} \in A$ and zero otherwise. That is, $N_n(A)$ is just the number of points of N_n contained in A.