Journal of Multivariate Analysis 123 (2014) 111–128

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Structure of the random measure associated with an isotropic stationary process

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ARTICLE INFO

Article history:
Received 14 June 2012
Available online 20 August 2013

AMS subject classification:
60G57
60G10
60B15
60H05

Keywords:
Random measures
Stationary processes
Tensor products
Isotropy
Spectral measures

ABSTRACT

Each stationary process can be biunivoquely associated with a random measure, through the Fourier transform. Consequently, every particularity of a process in the temporal domain has its corresponding one in the frequency domain. We propose to study the characteristics of the random measure when the process is isotropic. For that purpose, we will define the tensor product of random measures. A simulated example will illustrate such processes.

0. Introduction

Many natural phenomena are modelized with stationary isotropic processes, i.e. processes for which the covariance function does not depend on the direction. More precisely, such a process $(X_t)_{t\in\mathbb{R}^k}$ verifies $\langle X_t, X_{t'}\rangle = \langle X_{t-t'}, X_0\rangle = f(\|t-t'\|)$. These processes have been studied by numerous authors, such as Adler [1], Crujeiras et al. [6], Cucala et al. [7], Stein [11] and Yadrenko [12].

Any stationary process $(X_t)_{t\in\mathbb{R}^k}$ is the Fourier transform of a random measure Z (that is to say $X_t = \int e^{i\lambda t} dZ(\lambda)$), sometimes called a spectral representation. Therefore, any characteristic of a process in the time domain can be expressed in the frequency domain. The originality of this paper is to focus on the study of the random measure whose Fourier transform is an isotropic process, while most of existing papers that study isotropy focus on the study of the covariance function.

We show that the considered random measure Z is, according to a transformation, similar to the product of two classical measures. For this purpose, we have to define and study the tensor product $Z_1 \otimes Z_2$ of two random measures. This is done in a very general context, and represents an extension of the work already done by Boudou and Romain [4]. We study the properties of the tensor product $Z_1 \otimes Z_2$. In particular, we pay attention to the integration with respect to a measure of this type. Some of the results are close to Fubini Theorem, such as the possibility to swap the order of the integration.

In the particular case of isotropy, the random measures Z_1 and Z_2 are respectively defined on B, the unit sphere of $\mathbb{R}^k$, and on $\mathbb{R}_+^*$. We recall that $\mathbb{R}^k - \{0\}$ is homeomorph to $B \times \mathbb{R}_+^*$, a property which plays an important role here. We obtain different decompositions of the process thanks to the property of isotropy. We show properties such as that of invariance by

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0047-259X/$ – see front matter
http://dx.doi.org/10.1016/j.jmva.2013.08.001

projection of the process when its index moves on a sphere. We obtain, in a simple way, a technique for generating simulated isotropic processes. This can be useful for estimating the parameters of an isotropic process close to an observed process.

The first section is devoted to recalls. In the second section, we study the restriction of a measure, in order to adapt our study to the specificity of isotropic processes, for which the frequency 0 can be ignored. In Section 3, we examine three equivalent definitions of isotropy and recall the known properties of isotropic measures defined on the Borelian σ-field of $\mathbb{R}^k - \{0\}$. In Section 4, we study in a very general sense the tensor product of two random measures. In Section 5, we apply the results of the previous sections to the spectral measure of a stationary isotropic process. The last section is devoted to the expression of an isotropic process. We give different examples, and an illustration by simulation for a particular case.

1. Notation and recalls

In the all text, the equivalence class of an element φ belonging to an $\mathcal{L}^p$-type space is denoted $\widetilde{\varphi}$, and then belongs to an L^p-type space.

A random measure (r.m.) Z defined on $\mathcal{X}$, σ-field of parts of a set X, with values in a separable $\mathbb{C}$-Hilbert space H, is a vector measure such that, for any pair (A, B) of disjoint elements of $\mathcal{X}$, $\langle Z(A), Z(B)\rangle = 0$. Sometimes, Z is called a random field with orthogonal increments. Then it can be proved that the application $\mu_Z : A \in \mathcal{X} \mapsto \|Z(A)\|^2 \in \mathbb{R}_+$ is a bounded measure called a spectral measure (s.m.) of the r.m. Z. It is easy to verify that there exists one and only one isometry from $L^2(X, \mathcal{X}, \mu_Z)$ onto $H_Z = \overline{\text{vect}}\{Z(A), A \in \mathcal{X}\}$, which, with $\widetilde{1_A}$, associates $Z(A)$, for any $A \in \mathcal{X}$. The image by this isometry of an element $\widetilde{\varphi}$ of $L^2(X, \mathcal{X}, \mu_Z)$, denoted $\int \widetilde{\varphi}\mathrm{d}Z$ (sometimes, $\int \widetilde{\varphi}\mathrm{d}Z$ will be denoted $\int \varphi(x)\mathrm{d}Z(x)$), is called the integral of $\widetilde{\varphi}$ with respect to the r.m. Z.

If $(X', \mathcal{X}')$ is a second measurable space and f a measurable application from X into X', then the application $f(Z) : A' \in \mathcal{X}' \mapsto Z(f^{-1}(A')) \in H$ is a r.m., called an image of Z by f. Its spectral measure is the image $f(\mu_Z)$ of the bounded measure μ_Z by f. When φ' is an element of $\mathcal{L}^2(X', \mathcal{X}', \mu_{f(Z)})$, then $\varphi' \circ f$ belongs to $\mathcal{L}^2(X, \mathcal{X}, \mu_Z)$ and $\int \widetilde{\varphi'}\mathrm{d}f(Z) = \int \widetilde{\varphi' \circ f}\mathrm{d}Z$.

If J is a linear (resp. anti-linear) application from H into H', which preserves the norm, H' being a second separable Hilbert space, then

(a) $J \circ Z$ is a r.m., defined on $\mathcal{X}$ with values in H', whose spectral measure is μ_Z;
(b) for any φ of $\mathcal{L}^2(\mu_Z) = \mathcal{L}^2(\mu_{J\circ Z})$, we have $\int \widetilde{\varphi}\mathrm{d}J \circ Z = J(\int \widetilde{\varphi}\mathrm{d}Z)$ (resp. $\int \widetilde{\varphi}\mathrm{d}J \circ Z = J(\int \widetilde{\overline{\varphi}}\mathrm{d}Z)$).

A projector-valued spectral measure (p.v.s.m.) $\mathcal{E}$ on $\mathcal{X}$ for H is an application from $\mathcal{X}$ into $\mathcal{P}(H)$, set of the orthogonal projectors on H, such that

(a) $\mathcal{E}(X) = I_H$,
(b) $\mathcal{E}(A \cup B) = \mathcal{E}(A) + \mathcal{E}(B)$, for any pair of disjoint elements of $\mathcal{X}$,
(c) $\lim_n \mathcal{E}(A_n)(h) = 0$, for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{X}$ which decreasingly converges to $\emptyset$ and for any h of H.

We show (cf Boudou and Romain [3]) that for any r.m. Z, there is one, and only one, p.v.s.m. $\mathcal{E}$, on $\mathcal{X}$ for H_Z, called p.v.s.m. associated with Z, such that $\mathcal{E}(A)(\int \widetilde{\varphi}\mathrm{d}Z) = \int \widetilde{1_A\varphi}\mathrm{d}Z$, for any (A, φ) of $\mathcal{X} \times \mathcal{L}^2(X, \mathcal{X}, \mu_Z)$.

When k is an element of $\mathbb{N}^*$, $\mathbb{R}^k$ is endowed with the usual scalar product. A stationary continuous random function (c.r.f.) $(X_t)_{t\in\mathbb{R}^k}$, with values in H, is a family of elements of H such that:

(a) the application: $t \in \mathbb{R}^k \mapsto X_t \in H$ is continuous;
(b) for any pair (t_1, t_2) of elements of $\mathbb{R}^k$, we have $\langle X_{t_1}, X_{t_2}\rangle = \langle X_{t_1-t_2}, X_0\rangle$.

Of course, if H is of the type $L^2(\Omega, \mathcal{A}, P)$ and if $(X_t)_{t\in\mathbb{R}^k}$ is a centered process, we obtain the classical definition of the stationarity (because $\langle X_{t_1}, X_{t_2}\rangle = \mathrm{cov}(X_{t_1}, X_{t_2})$). Let $\mathcal{B}_{\mathbb{R}^k}$ be the Borel σ-field of $\mathbb{R}^k$. When Z is a r.m. defined on $\mathcal{B}_{\mathbb{R}^k}$, with values in H, then for any t of $\mathbb{R}^k$, the application $\mathrm{e}^{i\langle .,t\rangle} : x \in \mathbb{R}^k \mapsto \mathrm{e}^{i\langle x,t\rangle} \in \mathbb{C}$ is an element of $\mathcal{L}^2(\mathbb{R}^k, \mathcal{B}_{\mathbb{R}^k}, \mu_Z)$ and we show that $\left(\int \mathrm{e}^{i\langle x,t\rangle}\mathrm{d}Z(x)\right)_{t\in\mathbb{R}^k}$ is a stationary c.r.f. Conversely, if $(X_t)_{t\in\mathbb{R}^k}$ is a stationary c.r.f. with values in H, then there exists one, and only one, r.m. Z, called r.m. associated with $(X_t)_{t\in\mathbb{R}^k}$, such that $X_t = \int \mathrm{e}^{i\langle x,t\rangle}\mathrm{d}Z(x)$, for any t of $\mathbb{R}^k$. The measure μ_Z is also called the spectral measure of the stationary c.r.f. $(X_t)_{t\in\mathbb{R}^k}$.

When $(X_t)_{t\in\mathbb{R}^k}$ is a stationary c.r.f. with associated r.m. Z and when L is a linear application from $\mathbb{R}^p$ (p being an element of $\mathbb{N}^*$) into $\mathbb{R}^k$, then $(X_{L(t)})_{t\in\mathbb{R}^p}$ is a stationary c.r.f. of associated r.m. $L^*(Z)$. This last point is a particular case of a more general result that we can find developed in [2].

When $\{h'_n; n \in \mathbb{N}\}$ is an orthonormal basis of H', then, K being an element of $\sigma_2(H, H')$, the family $\{(K^*(h'_n)) \otimes h'_n; n \in \mathbb{N}\}$ is summable, of sum K.

Finally, when K is an element of $\sigma_2(H, H')$, h a normed element of H and h' a normed element of H', we can affirm that $h \otimes (K(h))$ (resp. $(K^*(h')) \otimes h'$) is the orthogonal projection of K onto the closed subspace $\{h \otimes u'; u' \in H'\}$ (resp. $\{u \otimes h'; u \in H\}$).

In order to simplify notation, when there is no ambiguity, we will omit the parentheses.

2. Integration with respect to a restricted measure

In this section, $(X, \mathcal{X})$ will be a measurable space, X' an element of $\mathcal{X}$, and $\mathcal{X}'$ the σ-field induced by $\mathcal{X}$ on X'. As X' belongs to $\mathcal{X}$, each element of $\mathcal{X}'$ is an element of $\mathcal{X}$, because of type $A \cap X'$, A belonging to $\mathcal{X}$. So we can define the restriction of a measure.