Journal of Multivariate Analysis 123 (2014) 143–159

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Strength of tail dependence based on conditional tail expectation

Lei Hua [a,*], Harry Joe [b]

[a] Division of Statistics, Northern Illinois University, DeKalb, IL, 60115, United States

[b] Department of Statistics, University of British Columbia, Vancouver, BC, V6T1Z4, Canada

ARTICLE INFO

Article history:
Received 1 October 2012
Available online 17 September 2013

AMS subject classification:
62H99

Keywords:
Tail order
Intermediate tail dependence
Tail quadrant independence
Stochastic increasing
Tail behavior
Copula
Boundary conditional distribution
Maximum domain of attraction

ABSTRACT

We use the conditional distribution and conditional expectation of one random variable given the other one being large to capture the strength of dependence in the tails of a bivariate random vector. We study the tail behavior of the boundary conditional cumulative distribution function (cdf) and two forms of conditional tail expectation (CTE) for various bivariate copula families. In general, for nonnegative dependence, there are three levels of strength of dependence in the tails according to the tail behavior of CTEs: asymptotically linear, sub-linear and constant. For each of these three levels, we investigate the tail behavior of CTEs for the marginal distributions belonging to maximum domain of attraction of Fréchet and Gumbel, respectively, and for copula families with different tail behavior.

1. Introduction

Tail behavior in terms of strength of dependence in the tails of a copula can have a lot of influence on inferences such as joint tail probability and risk assessment. In order to quantify the strength of dependence in the tails, one can use relevant quantities based on the limiting property of a copula through its diagonal. Let C be a bivariate copula and $(U, V) \sim C$ be a bivariate uniform random vector with C as the joint cumulative distribution function (cdf). Let $\overline{C}$ be the survival function of C, and $\widehat{C}$ be the associated survival copula. For a bivariate copula C, the lower tail dependence parameter is defined as $\lambda_L := \lim_{u\to 0^+} C(u,u)/u$, provided that the limit exists; similarly, the upper tail dependence parameter is defined as $\lambda_U := \lim_{u\to 0^+} \widehat{C}(u,u)/u$, provided that the limit exists. More generally [14] uses *tail order* as a measure of strength of dependence in the joint upper or joint lower tails. For the bivariate upper tail, the tail order κ is the exponent of u in the tail expansion

$$\overline{C}(1-u, 1-u) \sim u^{\kappa}\ell(u), \quad u \to 0^+,$$

where $\kappa \geq 1$, and ℓ is a slowly varying function (ℓ could be a constant). For a measurable function $g : \Re_+ \to \Re_+$, if for any constant $r > 0$, $\lim_{x\to 0^+} g(rx)/g(x) = 1$, then g is said to be slowly varying at 0^+, denoted as $g \in \mathrm{RV}_0(0^+)$; if for any constant $r > 0$, $\lim_{x\to\infty} g(rx)/g(x) = r^{\alpha}$, $\alpha \in \Re$, then g is said to be regularly varying at ∞ with variation exponent α, and is denoted as $g \in \mathrm{RV}_{\alpha}$. For a random variable X, when we say that X is regularly varying at ∞, it actually means that the

* Corresponding author.
E-mail addresses: hua@math.niu.edu (L. Hua), Harry.Joe@ubc.ca (H. Joe).

0047-259X/$ – see front matter
http://dx.doi.org/10.1016/j.jmva.2013.09.001

survival function $\overline{F} \in \mathrm{RV}_\alpha$ with some variation exponent $\alpha < 0$. There is an analogy with $C(u, u)$ in the joint lower tail. With this concept, smaller κ indicates more positive dependence in the tail: (a) $\kappa = 1$ for strongest dependence in the tail (usual tail dependence), (b) $1 < \kappa < 2$ which we call *intermediate tail dependence*, and (c) $\kappa = 2$ and slowly varying function ℓ with $\lim_{u\to 0^+} \ell(u) > 0$ which we call *tail quadrant or tail orthant independence*. It is possible for $\kappa > 2$ with negative dependence. The tail order is the reciprocal of the parameter η defined in [23], and used in the extreme value literature.

Another way to investigate strength of dependence in the tails is through the limiting property along the boundaries of the copula. Let $C_{2|1}(\cdot|u) = \partial C(u, v)/\partial u$ be the conditional cdf of $[V|U = u]$ and let $C_{1|2}(\cdot|v) = \partial C(u, v)/\partial v$ be the conditional cdf of $[U|V = v]$. Strength of dependence in the upper tail can be studied through the cdfs of $[V|U = 1]$ and $[U|V = 1]$, that is, the conditional cdfs $C_{2|1}(\cdot|1)$ and $C_{1|2}(\cdot|1)$. If $C_{2|1}(\cdot|1)$ is a degenerate distribution at 1 or has positive probability at the point 1, then this is an indication of relatively stronger positive dependence in the upper tail. Likewise, the behavior of $C_{2|1}(\cdot|0)$ and $C_{1|2}(\cdot|0)$ gives an indication of strength of dependence in the lower tail. The functions $C_{1|2}(\cdot|0)$, $C_{1|2}(\cdot|1)$, $C_{2|1}(\cdot|0)$ and $C_{2|1}(\cdot|1)$ are called *boundary conditional cdfs* in the remainder of this paper.

Depending on the value of $\vartheta_U = \vartheta_{U,1|2} := \lim_{u\to 1^-}\{1 - C_{1|2}(u|1)\}$ $[\vartheta_L = \vartheta_{L,1|2} := \lim_{u\to 0^+} C_{1|2}(u|0)]$, the strength of dependence in the upper [lower] tail can be (i) strongest if $\vartheta_U = 1$ $[\vartheta_L = 1]$; (ii) intermediate if $0 < \vartheta_U < 1$ $[0 < \vartheta_L < 1]$; (iii) weakest if $\vartheta_U = 0$ $[\vartheta_L = 0]$. For the upper tail, these correspond to (i) $C_{1|2}(\cdot|1)$ has mass of 1 at 1; (ii) $C_{1|2}(\cdot|1)$ has positive but not unit mass at 1; (iii) $C_{1|2}(\cdot|1)$ has no mass at 1; and similar interpretation holds for the lower tail. By symmetry, we have the same classification for $C_{2|1}(\cdot|1)$ and $C_{2|1}(\cdot|0)$. However, for permutation asymmetric bivariate copulas, the tail behavior of $C_{1|2}$ and $C_{2|1}$ may be different. For a copula family such as the Gumbel family with upper tail dependence (upper tail order $\kappa = 1$), we notice that the conditional cdf $C_{1|2}(\cdot|1)$ is degenerate at 1. The condition of $C_{1|2}(\cdot|1)$ being degenerate at 1 also came up in [7] in the analysis of tail dependence of sums of random variables. Properties of $C_{1|2}(\cdot|1)$ were also considered in [15] in the analysis of conditional tail expectation (CTE) of the form $\mathbb{E}[X_1|X_2 = t]$ and $\mathbb{E}[X_1|X_2 > t]$ as $t \to \infty$, where X_1, X_2 are dependent nonnegative random variables with copula C and a common univariate distribution F.

Although the concepts of tail order and boundary cdfs can both capture the strength of dependence in the tails, they are based on limiting properties of copulas along different routes, and there are no general relationships between them due to great flexibility of copulas. However, we will try to make connections between the tail order and the boundary conditional cdfs for parametric copula families, and for copulas under certain conditions when possible.

Finally, we consider using CTEs to investigate the strength of dependence in the tails. CTEs for strength of dependence of a bivariate dependent random vector play a similar role as mean excess functions for univariate tail heaviness. Mean excess functions of the form $\mathbb{E}[X - t|X > t]$ can be used to distinguish the strength of univariate tail heaviness (see Fig. 6.2.4 of [9]). Like the mean excess function, CTE of the form $\mathbb{E}[X_1|X_2 > t]$ can be easily approximated empirically. So the study of the tail behavior of such a CTE shall be helpful in developing relevant plots for visualizing the strength of dependence in the upper tails. Tail behavior of CTEs simply provides another way other than the usual tail dependence and tail order for looking at the strength of dependence in the tails. It is interesting to find connections between CTEs and the existing tail dependence concepts. However, the tail behavior of CTEs depends on both dependence structures and marginal distributions, the study here is much more complicated. For example, the usual tail dependence may lead to $\mathbb{E}[X_1|X_2 > t] = O(t)$ as $t \to \infty$ for F that has Pareto or power law tails with tail index $\alpha > 1$; see [35,13] for relevant references. For a copula family such as the Frank family with tail quadrant independence (tail order $\kappa = 2$), we notice that $\mathbb{E}[X_1|X_2 = t] = O(1)$ and $\mathbb{E}[X_1|X_2 > t] = O(1)$ as $t \to \infty$ for different F with finite mean, from direct calculations via Taylor expansions. We also have examples of intermediate tail dependence ($1 < \kappa < 2$) with $\mathbb{E}[X_1|X_2 = t] = O(t^\gamma)$ and $0 < \gamma < 1$, where we use standard techniques for asymptotic approximations of integrals.

We study different tail behavior in detail only for bivariate copulas. Since some multivariate dependence structures can be built up from bivariate copulas, the study on bivariate cases may provide guidance for models. For example, understanding the properties on strength of dependence in the tails for bivariate copulas is useful for (a) choosing bivariate linking copulas in vines, and (b) choosing bivariate copulas for consecutive observations in a Markov time series model. Models based on the pair-copula construction or vine copula are applied in [1,28]; a reference for use of copulas for Markov time series models is Chapter 8 of [19].

The paper is organized as the following. In Section 2, we establish possible combinations of tail order of form (a)–(c) and boundary conditional cdfs of form (i)–(iii). The form of the boundary conditional cdfs also affects how we proceed to approximate CTEs. In Section 2.1, a general relation can be derived under the assumption of a positive dependence condition called stochastically increasing (SI) and additional structures in the copula. Sections 2.2 and 2.3 have results for the classes of bivariate extreme value and Archimedean copulas, respectively. Then in Section 3, we obtain conditions for $\mathbb{E}[X_1|X_2 > t]$ and $\mathbb{E}[X_1|X_2 = t]$ to be asymptotically $O(t)$, $O(1)$ or $O(t^\gamma)$ where $0 < \gamma < 1$; we also refer to these three cases as asymptotically linear, constant and sub-linear, respectively. Finally, Section 4 concludes the paper with possible future research.

2. Boundary conditional cdfs of copulas

2.1. Overview

In this subsection, we discuss and prove some preliminary results with the positive dependence condition of SI.

The tail order for a bivariate copula involves only the upper corner near (1, 1) or the lower corner near (0, 0). However, if the horizontal line represents X_2 and the vertical line represents X_1, then the properties of $C_{2|1}(\cdot|1)$ depend on the copula