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Parameter estimation for operator scaling random fields

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ABSTRACT

Operator scaling random fields are useful for modeling physical phenomena with different scaling properties in each coordinate. This paper develops a general parameter estimation method for such fields which allows an arbitrary set of scaling axes. The method is based on a new approach to nonlinear regression with errors whose mean is not zero.

1. Introduction

Random fields are useful models for many natural phenomena (e.g., see Adler [1]). Self-similar random fields capture the fractal properties observed in applications (e.g., see Embrechts and Maejima [9]). An application to ground water hydrology laid out in Benson et al. [3] notes that the Hurst index of self-similarity can be expected to vary with the coordinate. In a two-dimensional model of an alluvial aquifer, a Hurst index $H_1 \geq 0.5$ models the organization of a porous medium in the natural direction of ground water flow, and another Hurst index $H_2 < 0.5$ describes negative dependence in the vertical direction, which captures the layering effect of the fluvial deposition process that created the medium structure. The scaling axes of the model often differ from the usual spatial coordinates. For example, there is often a dipping angle that tilts the first coordinate downward. In applications to fracture flow, a set of non-orthogonal scaling axes represents fracture orientations (e.g., see Ponson et al. [18] or Reeves et al. [19]).

To address such applications, Biermé et al. [6] developed a mathematical theory of operator scaling stable random fields (OSSRFs), based on ideas from [3]. An OSSRF is a scalar-valued random field $\{B(\boldsymbol{x})\}_{\boldsymbol{x}\in\mathbb{R}^d}$ such that

$$\{B(c^E\boldsymbol{x})\}_{\boldsymbol{x}\in\mathbb{R}^d} \triangleq \{cB(\boldsymbol{x})\}_{\boldsymbol{x}\in\mathbb{R}^d} \quad \text{for all } c > 0, \tag{1.1}$$

where E is a $d \times d$ scaling matrix whose eigenvalues have real part greater than zero, $c^E = \exp(E \log c)$, with $\exp(A) = I + A + A^2/2! + \cdots$ the usual matrix exponential, and $\triangleq$ denotes equality of all finite-dimensional distributions. If the scaling matrix E has a basis of eigenvectors $E\,\boldsymbol{b}_i = a_i\boldsymbol{b}_i$ for $i = 1, \ldots, d$, then $c^E\boldsymbol{b}_i = c^{a_i}\boldsymbol{b}_i$ for $i = 1, \ldots, d$, and it follows immediately from (1.1) that the one-dimensional slice $B_i(t) := B(t\boldsymbol{b}_i)$ is self-similar with Hurst index $H_i = 1/a_i$, i.e.,

$$\{B_i(ct)\}_{t\in\mathbb{R}} \triangleq \{c^{H_i}B_i(t)\}_{t\in\mathbb{R}} \quad \text{for all } c > 0.$$

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In particular, the Hurst index H_i of self-similarity varies with the coordinate, and the scaling axes $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d$ can be any basis for $\mathbb{R}^d$. The construction of the OSSRF in [6] ensures that the random field has stationary increments, i.e.,

$$\{B(\boldsymbol{x}+\boldsymbol{h}) - B(\boldsymbol{h})\}_{\boldsymbol{x}\in\mathbb{R}^d} \triangleq \{B(\boldsymbol{x})\}_{\boldsymbol{x}\in\mathbb{R}^d} \quad \text{for all } \boldsymbol{h} \in \mathbb{R}^d,$$

and then it follows that any one-dimensional slice $B_{\boldsymbol{x},i}(t) := B(\boldsymbol{x} + t\boldsymbol{b}_i) - B(\boldsymbol{x})$ is self-similar with Hurst index H_i. If the random field is Gaussian, then $B_i(t) := B(t\boldsymbol{b}_i)$ is a fractional Brownian motion with Hurst index $H_i = 1/a_i$, since this is the only self-similar Gaussian process with stationary increments [20, Corollary 7.2.3]. OSSRFs were applied to ground water hydrology by Hu et al. [12] to synthesize realistic porosity fields and hydraulic conductivity fields, consistent with aquifer data. The multi-scaling produces organized regions of high porosity (and/or conductivity) that create preferential flow paths, an important feature of realistic random field simulations that is not present in an isotropic model.

Practical applications of multi-scaling random fields require a method to estimate the parameters. For the special case where the scaling axes equal the original Euclidean coordinates, estimation methods have been developed by Beran et al. [4], Boissy et al. [7], and Guo et al. [10]. However, applications to geophysics require a more general approach, with an arbitrary set of scaling axes. This paper develops a general method for parameter estimation, which also estimates the appropriate scaling axes. These axes need not be orthogonal. Our approach is based on a new method for nonlinear regression with errors whose mean is not zero. This method for nonlinear regression may well have further applications in other areas.

In Section 2, we review OSSRFs and outline the parameter estimation problem, which involves a nonlinear regression where the errors do not have a zero mean. In Section 3, we propose a new nonlinear regression method to handle the nonzero mean error, and prove consistency and asymptotic normality for this estimator. In Section 4, we return to OSSRFs and apply the proposed nonlinear regression method to estimate parameters. Section 5 summarizes the results of a brief simulation study, to verify that the method gives reasonably accurate parameter estimates in practice. Some concluding remarks are contained in Section 6.

2. Operator scaling random fields

In this section, we recall the spectral method for constructing OSSRFs; see Biermé et al. [6] for complete details. Then we outline the proposed nonlinear regression method for parameter estimation.

Given a $d \times d$ scaling matrix E whose eigenvalues all have positive real part, we say that a continuous function $\psi : \mathbb{R}^d \to [0, \infty)$ is E^T-homogeneous if $\psi(c^{E^T}\boldsymbol{\xi}) = c \cdot \psi(\boldsymbol{\xi})$ for all $c > 0, \boldsymbol{\xi} \in \mathbb{R}^d$. Then Theorem 4.1 in [6] shows that there exists a stochastically continuous OSSRF

$$B(\boldsymbol{x}) = \mathrm{Re}\left[\int_{\boldsymbol{\xi}\in\mathbb{R}^d} \left(e^{i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} - 1\right)\psi(\boldsymbol{\xi})^{-1-q/\alpha} W_\alpha(d\boldsymbol{\xi})\right], \tag{2.1}$$

where $q = \mathrm{trace}(E)$, $\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle = \sum_{i=1}^d x_i\xi_i$ and $W_\alpha(d\boldsymbol{\xi})$ is a complex isotropic symmetric stable random measure with index $0 < \alpha \le 2$ and control measure $m(d\boldsymbol{\xi}) = \sigma_0^2 d\boldsymbol{\xi}$. If $\alpha = 2$, then $B(\boldsymbol{x})$ is a Gaussian random field, and for any Borel subset A of $\mathbb{R}^d$ we have $W_2(A) = Z_1 + iZ_2$, where Z_1 and Z_2 are independent and identically distributed (i.i.d.) Gaussian random variables on $\mathbb{R}^1$ with mean zero and variance $\sigma_0^2|A|/2$, so $\mathbb{E}[W_2(A)^2] = |A|$, the Lebesgue measure of A. Corollary 4.2 in [6] shows that the OSSRF (2.1) has stationary increments, and that the operator scaling property (1.1) holds. See for example Samorodnitsky and Taqqu [20] for general details on stable stochastic integrals.

Next, we review a spectral method for simulating the OSSRF (2.1), using a fast Fourier transform (FFT); see Kegel [15] for complete details. This method yields a spatial regression model for OSSRFs that is the basis for our parameter estimation scheme. To simplify the discussion, we focus on the case of Gaussian OSSRFs with $\alpha = 2$ in two dimensions. However, everything extends easily to stable OSSRFs on $\mathbb{R}^d$ with index $0 < \alpha < 2$. First, we approximate the stochastic integral in (2.1) by a Riemann sum. Let $\mathcal{D} = [-A, A]^2 \setminus (-B, B)^2 \subset \mathbb{R}^2$ be a large square centered at the origin with radius A, with a much smaller square of radius B deleted to form an annular region, such that B/A is a rational number. Select a large integer M such that $(B/A)M$ is also an integer. Next we subdivide the region $\mathcal{D}$ into small squares of size A/M. Define $\mathcal{I} = \{-M, \ldots, M-1\}^2$ and $\mathcal{J} = \mathcal{I} \setminus \{-(B/A)M, \ldots, (B/A)M - 1\}^2$, a collection of integer grid points in $\mathbb{R}^2$, and set $\boldsymbol{\xi}_{\boldsymbol{k}} = (A/M)\boldsymbol{k}$ for $\boldsymbol{k} = (k_1, k_2) \in \mathcal{J}$. Now let $\Delta\boldsymbol{\xi}_{\boldsymbol{k}}$ be the square of side A/M with the point $\boldsymbol{\xi}_{\boldsymbol{k}}$ at its southwest corner, i.e., $\Delta\boldsymbol{\xi}_{\boldsymbol{k}} = [(A/M)k_1, (A/M)(k_1+1)] \times [(A/M)k_2, (A/M)(k_2+1)]$. Then we define

$$J_{\mathcal{D}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{J}} \left(e^{i\langle \boldsymbol{x},\boldsymbol{\xi}_{\boldsymbol{k}}\rangle} - 1\right)\psi(\boldsymbol{\xi}_{\boldsymbol{k}})^{-1-q/2} W_2(\Delta\boldsymbol{\xi}_{\boldsymbol{k}}), \tag{2.2}$$

where the complex-valued random variables $W_2(\Delta\boldsymbol{\xi}_i)$ are i.i.d. with $(\sigma_0 A/M)(Z_1 + iZ_2)$, and Z_1 and Z_2 are i.i.d. $\mathcal{N}(0, 1/2)$. As $M \to \infty$, the approximating sum $J_{\mathcal{D}}(\boldsymbol{x})$ converges in probability to the stochastic integral

$$I_{\mathcal{D}}(\boldsymbol{x}) = \int_{\boldsymbol{\xi}\in\mathcal{D}} \left(e^{i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} - 1\right)\psi(\boldsymbol{\xi})^{-1-q/2}\, W_2(d\boldsymbol{\xi}),$$

since the integrand is continuous on the compact set $\mathcal{D}$ (e.g., see [16, Section 7.7]). Since the stochastic integral (2.1) exists, $I_{\mathcal{D}}(\boldsymbol{x})$ converges in probability to (2.1) as $A \to \infty$ and $B \to 0$.