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On compatibility of discrete full conditional distributions: A graphical representation approach

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ABSTRACT

To deal with the compatibility issue of full conditional distributions of a (discrete) random vector, a graphical representation is introduced where a vertex corresponds to a configuration of the random vector and an edge connects two vertices if and only if the ratio of the probabilities of the two corresponding configurations is specified through one of the given full conditional distributions. Compatibility of the given full conditional distributions is equivalent to compatibility of the set of all specified probability ratios (called the ratio set) in the graphical representation. Characterizations of compatibility of the ratio set are presented. When the ratio set is compatible, the family of all probability distributions satisfying the specified probability ratios is shown to be the set of convex combinations of k probability distributions where k is the number of components of the underlying graph.
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1. Introduction

The problem of characterizing a joint probability distribution by conditionals has been extensively studied in the last few decades. Consider a set of n real-valued random variables $\underline{X} = (X_1, \ldots, X_n)$ whose joint distribution is to be determined. Given some conditional distributions of the form $p_{S|T}(\underline{x}_S|\underline{x}_T)$ (the conditional probability mass/density function of $(X_i)_{i\in S}$ at $\underline{x}_S$ given $(X_i)_{i\in T} = \underline{x}_T$), where S ($\neq\emptyset$) and T are disjoint subsets of $\{1, \ldots, n\}$, it is desired (i) to determine whether these conditionals are compatible in the sense that they are the conditional distributions of some joint distribution, and (ii) to find all such (compatible) joint distributions when the given conditionals are compatible. See [1,2,6,10,11] for general results and comprehensive discussions. See also [4,13–18] for more recent developments. (It should be noted that most of the above papers consider the case where the given conditionals of the form $p_{S|T}$ are full in the sense that $S \cup T = \{1, \ldots, n\}$.) In case that $X_1, \ldots, X_n$ refer to certain observations at n locations in a region, the problem falls in the area of spatial statistics. In particular, when $X_1, \ldots, X_n$ form a Markov random field, the famous Hammersley–Clifford theorem characterizes the joint distribution via the Gibbs measure, cf. [7].

In the present paper, we restrict attention to the case where each X_i takes values in a finite set and the given conditionals are full. Note that specifying a full conditional $p_{S|T}$ amounts to specifying the probability ratio $p(\underline{x})/p(\underline{x}')$ for all $\underline{x} = (\underline{x}_S, \underline{x}_T)$ and $\underline{x}' = (\underline{x}'_S, \underline{x}'_T)$ with $\underline{x}_T = \underline{x}'_T$ where $\underline{x}_S$ denotes $\underline{x}$ restricted to the subset S of $\{1, \ldots, n\}$. With this simple observation, in the next section, we reformulate the problem more generally in terms of a graphical representation where each vertex corresponds to a configuration of $\underline{X}$ and an edge connects two vertices if and only if the ratio of the probabilities of the two corresponding configurations is specified. (It should be remarked that the graphical representation introduced here is different

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from the graph for a Markov random field where a vertex (usually called a *site*) corresponds to a random variable in the Markov random field and an edge connects two sites (called *neighbors*) if the two corresponding random variables have *local interactions*; see e.g. [7] for the precise definitions of technical terms in Markov random fields.) In Section 3, results on compatibility of the specified probability ratios in a graphical representation are presented. Furthermore, all compatible joint distributions are characterized when the specified probability ratios are compatible, in which case there is a unique compatible joint distribution if and only if the underlying graph is connected. Section 4 presents a simple algorithm for checking compatibility of the specified probability ratios in a graphical representation. Section 5 contains concluding remarks.

2. Graphical representation

Consider a graph with vertex set V and edge set E where an edge connecting vertices $u, v \in V$ is denoted by $\{u, v\}$ (so that E is identified with a subset of the collection of all 2-element subsets of V). See e.g. [8] for an introduction to graph theory. For each edge $\{u, v\}$, there is a specified ratio $r(u, v) : r(v, u)$ where $r(u, v)$ and $r(v, u)$ are positive numbers. Let $R = R(E)$ denote the collection of all the specified ratios (to be called the ratio set), and we refer to (V, E, R) as a graphical representation. It is desired (i) to determine whether R is compatible in the sense that there is a (positive) probability distribution $(p(v))_{v\in V}$ on the vertex set V such that $p(u) : p(v) = r(u, v) : r(v, u)$ for all $\{u, v\} \in E$, and (ii) to find all (compatible) probability distributions satisfying R when R is compatible.

Let $\mathscr{C} = \{p_{S|T}\}$ be a given set of full conditionals for n discrete random variables $\underline{X} = (X_1, \ldots, X_n)$ taking values in $\mathscr{X} = \mathscr{X}_1 \times \cdots \times \mathscr{X}_n$. If $p_{S|T}$ and $p_{S'|T'} \in \mathscr{C}$ are conditionals of some joint distribution p and if $p_{S'|T'}(\underline{x}^0_{S'}|\underline{x}^0_{T'}) = 0$ for some $\underline{x}^0 \in \mathscr{X}$, then $p(\underline{x}^0)$ is necessarily zero, which in turn implies $p_{S|T}(\underline{x}^0_S|\underline{x}^0_T) = 0$. Consequently, a necessary condition for $\mathscr{C}$ to be compatible is that if $\underline{x}^0 \in \mathscr{X}$ is such that $p_{S'|T'}(\underline{x}^0_{S'}|\underline{x}^0_{T'}) = 0$ for some $p_{S'|T'} \in \mathscr{C}$, then $p_{S|T}(\underline{x}^0_S|\underline{x}^0_T) = 0$ for all $p_{S|T} \in \mathscr{C}$. We will refer to this necessary condition as condition (C1). Letting

$$O_{\mathscr{C}} := \{\underline{x} \in \mathscr{X} : p_{S|T}(\underline{x}_S|\underline{x}_T) = 0 \text{ for some } p_{S|T} \in \mathscr{C}\},$$

condition (C1) is equivalent to

$$O_{\mathscr{C}} = \{\underline{x} \in \mathscr{X} : p_{S|T}(\underline{x}_S|\underline{x}_T) = 0 \text{ for all } p_{S|T} \in \mathscr{C}\}.$$

Note that condition (C1) can also be stated as: "For each $p_{S|T} \in \mathscr{C}$, the set $O(p_{S|T}) := \{\underline{x} \in \mathscr{X} : p_{S|T}(\underline{x}_S|\underline{x}_T) = 0\}$ is the same for all $p_{S|T} \in \mathscr{C}$". Theorem 1.1(i) of [1] gives this necessary condition for $n = 2$ and $\mathscr{C} = \{p_{\{1\}|\{2\}}, p_{\{2\}|\{1\}}\}$. In what follows, we will always assume that $\mathscr{C}$ satisfies condition (C1). Define $V_{\mathscr{C}} = \mathscr{X} \setminus O_{\mathscr{C}}$. Note that if all $p_{S|T} \in \mathscr{C}$ are conditionals of some joint distribution p (implying that $\mathscr{C}$ is compatible), then p has $V_{\mathscr{C}}$ as its support.

We now define the graphical representation for $\mathscr{C}$. Let $V_{\mathscr{C}}$ be the vertex set. An edge $\{\underline{x}, \underline{x}'\}$ connects two vertices $\underline{x}, \underline{x}' \in V_{\mathscr{C}}$ if and only if $\underline{x}_T = \underline{x}'_T$ for some $p_{S|T} \in \mathscr{C}$; the ratio associated with this edge is given by

$$r(\underline{x}, \underline{x}') : r(\underline{x}', \underline{x}) = p_{S|T}(\underline{x}_S|\underline{x}_T) : p_{S|T}(\underline{x}'_S|\underline{x}'_T).$$

The resulting graphical representation is denoted by $(V_{\mathscr{C}}, E_{\mathscr{C}}, R_{\mathscr{C}})$. It should be noted that an edge may connect two vertices $\underline{x}$ and $\underline{x}' \in V_{\mathscr{C}}$ through two or more conditionals in $\mathscr{C}$, resulting in possibly different ratios associated with this edge. For example, consider $n = 3$ and $\mathscr{C} = \{p_{\{1,2\}|\{3\}}, p_{\{1,3\}|\{2\}}\}$. Then an edge connects $\underline{x} = (x_1, x_2, x_3)$ and $\underline{x}' = (x'_1, x_2, x_3)$ (with $x_1 \neq x'_1$) through either of the two conditionals in $\mathscr{C}$, resulting in two ratios given by

$$p_{\{1,2\}|\{3\}}((x_1, x_2)|x_3) : p_{\{1,2\}|\{3\}}((x'_1, x_2)|x_3) \quad \text{and} \quad p_{\{1,3\}|\{2\}}((x_1, x_3)|x_2) : p_{\{1,3\}|\{2\}}((x'_1, x_3)|x_2).$$

(Such cases cannot happen if S is a singleton for every $p_{S|T} \in \mathscr{C}$.) Clearly in order for $\mathscr{C}$ to be compatible, the ratios associated with an edge are necessarily equal if the edge is formed through two or more conditionals in $\mathscr{C}$. Below we will also assume that $\mathscr{C}$ satisfies this (second) necessary condition (to be referred to as condition (C2)), so that exactly one ratio is associated with each edge. Note that condition (C2) is trivially satisfied if S is a singleton for every $p_{S|T} \in \mathscr{C}$. The following theorem shows that compatibility of $\mathscr{C}$ is equivalent to compatibility of $R_{\mathscr{C}}$.

Theorem 1. *Assume that $\mathscr{C}$ satisfies conditions* (C1) *and* (C2)*. Then a joint distribution with support $V_{\mathscr{C}}$ satisfies $\mathscr{C}$ if and only if it satisfies $R_{\mathscr{C}}$. Consequently, $\mathscr{C}$ is compatible if and only if $R_{\mathscr{C}}$ is compatible.*

Proof. To prove the "only if" part, suppose that a joint distribution p (with support $V_{\mathscr{C}}$) satisfies $\mathscr{C}$, i.e. under p the conditional distribution of $\underline{X}_S$ given $\underline{X}_T$ agrees with $p_{S|T}$ for every $p_{S|T} \in \mathscr{C}$. Consider an (arbitrary) edge $\{\underline{x}, \underline{x}'\} \in E_{\mathscr{C}}$ with an associated ratio given by $r(\underline{x}, \underline{x}') : r(\underline{x}', \underline{x}) = p_{S|T}(\underline{x}_S|\underline{x}_T) : p_{S|T}(\underline{x}'_S|\underline{x}'_T)$ (for some $p_{S|T} \in \mathscr{C}$). By definition, we have $\underline{x}_T = \underline{x}'_T$. Then p satisfies the associated ratio specification since

$$\frac{p(\underline{x})}{p(\underline{x}')} = \frac{p_{S|T}(\underline{x}_S|\underline{x}_T)}{p_{S|T}(\underline{x}'_S|\underline{x}'_T)} = \frac{r(\underline{x}, \underline{x}')}{r(\underline{x}', \underline{x})},$$

from which it follows that p satisfies $R_{\mathscr{C}}$.

To prove the "if" part of the theorem, suppose that a joint distribution p (with support $V_{\mathscr{C}}$) satisfies $R_{\mathscr{C}}$. Consider a conditional $p_{S|T} \in \mathscr{C}$. Fix an (arbitrary) $\underline{x}^0 \in V_{\mathscr{C}}$. For every $\underline{x} \in V_{\mathscr{C}}$ with $\underline{x}_T = \underline{x}^0_T$, we have (by definition) $\{\underline{x}^0, \underline{x}\} \in E_{\mathscr{C}}$ and $r(\underline{x}^0, \underline{x}) : r(\underline{x}, \underline{x}^0) = p_{S|T}(\underline{x}^0_S|\underline{x}^0_T) : p_{S|T}(\underline{x}_S|\underline{x}_T)$. It follows that

$$\frac{p(\underline{x})}{p(\underline{x}^0)} = \frac{r(\underline{x}, \underline{x}^0)}{r(\underline{x}^0, \underline{x})} = \frac{p_{S|T}(\underline{x}_S|\underline{x}_T)}{p_{S|T}(\underline{x}^0_S|\underline{x}^0_T)} = \frac{p_{S|T}(\underline{x}_S|\underline{x}^0_T)}{p_{S|T}(\underline{x}^0_S|\underline{x}^0_T)}.$$

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