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Admissibility of linear estimators for the stochastic regression coefficient in a general Gauss–Markoff model under a balanced loss function

Mingxiang Cao *

Department of Mathematics, Hefei Normal University, Hefei 230601, PR China
Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China

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ABSTRACT

In this paper, problems of linearly admissible estimators for stochastic regression coefficients are considered in a general Gauss–Markoff model with random effects. The generalized balanced loss function is given, and under it the admissibility of linear estimators is investigated. Sufficient and necessary conditions for linear estimators to be admissible in classes of homogeneous and inhomogeneous linear estimators are obtained, separately.

1. Introduction

We open this section with some notation. For a matrix A, $\mu(A)$, A', A^-, A^+, $rk(A)$ and $tr(A)$ denote, respectively, the range space, transpose, g-inverse, Moore–Penrose inverse, rank and trace. The $n \times n$ identity matrix is denoted by I_n. For nonnegative definite matrices A and B, $A \geq B$ and $A > B$ stand for the nonnegativity and positivity of the matrix $A - B$, respectively. The set of all real $m \times n$ matrices is denoted by $\mathcal{R}^{m\times n}$.

The problems of linear admissible estimators in linear models have been extensively examined in the literature under some classical loss functions, such as the quadratic loss, the vector loss and the matrix loss, considering only errors of estimation (see [14,10,17,5,21] etc.). [20] proposed a balanced loss function that takes the error of estimation and goodness of fit into account. Hence, compared to classical loss functions, it is a more comprehensive and reasonable standard for measuring the estimates. Moreover, it is worth mentioning that the balanced loss function is more sensitive than the quadratic loss function, which means that if an estimator is admissible under the balanced loss, it is also admissible under the quadratic loss. The sensitivity of the quadratic loss function lies between those of the vector loss function and the matrix loss function. Therefore, the problems of estimation studied under the balanced loss function are significant. For more details, the readers are referred to [15,16,2,3,11,12,19,6,7,9,1,8]. Appreciating the popularity of the balanced loss function, we extend it further and present a general loss function for the following model in the paper.

Throughout this article a general Gauss–Markoff random effects model of the form

$$Y = X\beta + e \tag{1.1}$$

* Correspondence to: Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China.
E-mail address: caomingx@163.com.

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will be considered. The Y is an n-dimensional vector of observations, X is an $n\times p$ known matrix of rank p, β is a p-dimensional vector of stochastic regression coefficients and e is an n-dimensional error vector, where $\mathrm{E}(e)=0$, $Cov(e)=\sigma^2 V$, $\mathrm{E}(\beta)=K\theta$, $Cov(\beta)=\sigma^2\Lambda$, $\mathrm{E}(\beta e')=0$. Furthermore, $\Lambda, V\geq 0 (V\neq 0)$ and $K_{p\times r}$ are known matrices. In order to avoid unidentifiability, we assume that $rk(K)=r$. Nevertheless $\theta\in\mathcal{R}^r$ and $\sigma^2>0$ are unknown parameters. We easily get $Cov(Y)=\sigma^2(V+X\Lambda X')$. When $\Lambda=0$, the β is non-stochastic. When the (i,i) element of Λ is zero, the ith element of β is also zero, and in this case some elements of β are stochastic and others are non-stochastic. Therefore, model (1.1) has extensive backgrounds and applications.

For model (1.1), according to Zellner's idea of balanced loss and the unified theory of least squares (see [14]), we propose the generalized balanced loss function

$$L(d,\beta)=w(Y-Xd)'T^-(Y-Xd)+(1-w)(d-\beta)'S(d-\beta), \tag{1.2}$$

where w is a scalar lying between 0 and 1 which provides the weight assigned to the goodness of fit of the model. The matrix $S>0$ is known, $d=d(Y)$ is an estimator of β and $T=V+XUX'$ with $rk(T)=rk(V,X)$ for arbitrary matrix $U\geq 0$. For U, without loss of generality, we select $U=I$. We get that (1.2) is unrelated to the selection of T^-. The corresponding risk function is defined by

$$R(d;\theta,\sigma^2):=\mathrm{E}L(d,\beta).$$

For model (1.1), [13] studied extension of the Gauss–Markoff theorem, [6,7] investigated the efficiency of shrinkage estimators under loss function (1.2) when $V=I$ and $U=0$, and [4] researched the admissibility of linearly combined estimators of β and θ under the quadratic loss function. However, not too much work has been carried out on the admissibility of the stochastic regression coefficient with respect to a balanced loss function.

In this paper, a study of the problem of admissibility for linear estimators of the stochastic regression coefficient β in model (1.1) under balanced loss function (1.2) is given in a general situation, and the sufficient and necessary conditions for linear estimators to be admissible in $\mathcal{LH}$ and $\mathcal{LI}$ are obtained, separately, where $\mathcal{LH}=\{LY: L\in\mathcal{R}^{p\times n}\}$ and $\mathcal{LI}=\{LY+\alpha: L\in\mathcal{R}^{p\times n} \text{ and } \alpha\in\mathcal{R}^p\}$ denote the classes of homogeneous and inhomogeneous linear estimators, respectively. These results contain the main results of [19,8].

The organization of this paper is as follows. In Section 2, we give our main results, namely the sufficient and necessary conditions for linear estimators of stochastic regression coefficient to be admissible, some corollaries and remarks. Section 3 contains some definitions and lemmas playing important roles in the paper. Section 4 shows proofs of the main results. Conclusions are assigned in Section 5.

2. The main results

Theorem 2.1. *Under model (1.1) and balanced loss function (1.2),* $LY\overset{\mathcal{LH}}{\sim}\beta$ *holds if and only if*

(a) $LV=LX(X'T^-X)^{-1}X'T^-V$,
(b) $\{\widehat{LX}-(1-w)B^{-1}S\Lambda[KK'+(X'T^-X)^{-1}+\Lambda-I]^+\}(I-F)[(X'T^-X)^{-1}+\Lambda-I]=0$,
(c) $\widehat{L}XF[(X'T^-X)^{-1}+\Lambda-I]F'X'\widehat{L}'\leq(1-w)\{\widehat{L}XF[(X'T^-X)^{-1}+\Lambda-I]F'SB^{-1}+B^{-1}S\Lambda F'(LX-I)'\}$, *and*
(d) $\mu[(LX-I)K]=\mu[(LX-I)F(X'T^-X-I)]$

hold simultaneously, where $F=K\{K'[KK'+(X'T^-X)^{-1}+\Lambda-I]^+K\}^-K'[KK'+(X'T^-X)^{-1}+\Lambda-I]^+$, $\widehat{L}=L-wB^{-1}X'T^-$ *and* $B=wX'T^-X+(1-w)S$.

Let (e) $rk[(LX-I)K]=rk[(LX-I)F(X'T^-X-I)]$; then we can get an expression equivalent to Theorem 2.1, which is given as follows.

Theorem 2.2. *Under model (1.1) and balanced loss function (1.2),* $LY\overset{\mathcal{LH}}{\sim}\beta$ *holds if and only if* (a)–(c) *and* (e) *hold simultaneously.*

If $K=I$, then the simple result of Theorem 2.1 can be obtained.

Corollary 2.1. *Under model (1.1) with* $K=I$ *and balanced loss function (1.2),* $LY\overset{\mathcal{LH}}{\sim}\beta$ *holds if and only if* (a),
(c′) $\widehat{L}X[(X'T^-X)^{-1}+\Lambda-I]X'\widehat{L}'\leq(1-w)\{\widehat{L}X[(X'T^-X)^{-1}+\Lambda-I]SB^{-1}+B^{-1}S\Lambda(LX-I)'\}$ *and*
(d′) $rk(LX-I)=rk[(LX-I)(X'T^-X-I)]$

hold simultaneously.

When $V>0$ and $\Lambda\neq 0$, (1.1) is called the nonsingular random effects model. This model is more often used in practical problems than singular random effects model (1.1), which often appears in theoretical studies. For the nonsingular model, we can select $U=0$. Then the balanced loss function (1.2) turns into the following:

$$L(d,\beta)=w(Y-Xd)'V^{-1}(Y-Xd)+(1-w)(d-\beta)'S(d-\beta). \tag{2.1}$$

And the corresponding result is as follows.