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Stochastic comparisons of order statistics and their concomitants

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ABSTRACT

Let $X_{1:n} \le X_{2:n} \cdots \le X_{n:n}$ be the order statistics from some sample, and let $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ be the corresponding concomitants. One purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ to the random vector $(X_{r+1:n}, Y_{[r+1:n]})$, $r = 1, 2, \ldots, n-1$. Such comparisons are called *one-sample comparisons*. Next, let $S_{1:n} \le S_{2:n} \cdots \le S_{n:n}$ be the order statistics constructed from another sample, and let $T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}$ be the corresponding concomitants. Another purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ with the random vector $(S_{r:n}, T_{[r:n]})$, $r = 1, 2, \ldots, n$. Such comparisons are called *two-sample comparisons*. It is shown that some of the results in this paper strengthen previous results in the literature. Some applications in reliability theory are described.

1. Introduction

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent copies of a random vector (X, Y). Let $X_{1:n} \le X_{2:n} \cdots \le X_{n:n}$ be the order statistics constructed from the sample of the first coordinates $X_1, X_2, \ldots, X_n$. Denote the Y-variate associated with $X_{i:n}$ by $Y_{[i:n]}$, $i = 1, 2, \ldots, n$; that is, $Y_{[i:n]} = Y_k$ if, and only if, $X_{i:n} = X_k$. The random variables $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ are called the concomitants of the order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. One purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ to the random vector $(X_{r+1:n}, Y_{[r+1:n]})$, $r = 1, 2, \ldots, n-1$. Such comparisons will be called below *one-sample comparisons*.

Next, let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of a random vector, but this time that random vector is (S, T), which generally has a different distribution function than (X, Y). Let $S_{1:n} \le S_{2:n} \cdots \le S_{n:n}$ be the order statistics constructed from the sample of the first coordinates $S_1, S_2, \ldots, S_n$, and let $T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}$ be the associated concomitants. A purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ with the random vector $(S_{r:n}, T_{[r:n]})$, $r = 1, 2, \ldots, n$. Such comparisons will be called below *two-sample comparisons*.

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Applications of concomitants of order statistics can be found in many areas of probability and statistics. For example, concomitants are of interest in a variety of estimation problems (see [8]), in selection and prediction problems (see [16]), in insurance (see [5]), and in reliability theory (see [2]).

Some papers that studied positive dependence and/or stochastic orders involving concomitants of order statistics are the following. Khaledi and Kochar [16] identified conditions on the distribution function of (X, Y) under which the concomitants are ordered with respect to several univariate stochastic orders, whereas Khaledi and Kochar [16] and Blessinger [9] identified conditions on the distribution function of (X, Y) under which the concomitants are positively dependent in various senses. Eryilmaz [10] obtained results that stochastically compare concomitants without assuming that $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are identically distributed. Izadi and Khaledi [11] considered stochastic orderings of concomitants of progressive type II censored order statistics, and recently Amini et al. [1] studied properties and orderings of concomitants of record values. A study of concomitants of order statistics for dependent samples was developed by [22].

In this paper "increasing" and "decreasing" stand for "nondecreasing" and "nonincreasing", respectively. For any random variable (or vector) Z and an event A, we will denote by $[Z\big|A]$ any random variable that is distributed according to the conditional distribution function of Z given A. We will use the notation $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

In Sections 2 and 3 below we describe, respectively, our one-sample and two-sample comparison results. In Section 4 we mention examples and applications of the technical results, and we make various remarks on their relations to other results in the literature, and on strictness of the assumptions in different results. But before proceeding to the technical results in Sections 2 and 3 we list below and make some comments on various stochastic orders that will be used in the sequel.

All the stochastic orders that are mentioned in this paper can be found in [19] or in [21]. Specifically, $\leq_{\rm st}$ stands for the univariate *ordinary stochastic order* when two univariate random variables are compared (see, for example, [21, page 3]), and it stands for the multivariate ordinary stochastic order when two random vectors are compared (see its formal definition in [21, page 266]). The notation $\leq_{\rm hr}$ stands for the univariate *hazard rate order* when two univariate random variables are compared (see, for example, [21, page 16]), and it stands for the multivariate hazard rate order when two random vectors are compared (see its formal definition in (6.D.1) in [21]). Another order that will be used in this paper is the *likelihood ratio order* $\leq_{\rm lr}$. The univariate definition of it can be found in [21, page 42], whereas the multivariate definition is given in that reference in page 298. Recall also that for two random vectors (X, Y) and (S, T) we say that (X, Y) is smaller than (S, T) in the *upper orthant order*, denoted as $(X, Y) \leq_{\rm uo} (S, T)$ (respectively, (X, Y) is smaller than (S, T) in the *lower orthant order*, denoted as $(X, Y) \leq_{\rm lo} (S, T)$) if $P\{X > x, Y > y\} \leq P\{S > x, T > y\}$ (respectively, $P\{X \leq x, Y \leq y\} \leq P\{S \leq x, T \leq y\}$) for all $(x, y) \in \mathbb{R}^2$; see Section 6.G.1 in [21].

We further recall the definition of the *strong stochastic order* (see [18] or [21, page 268]). Let (X, Y) and (S, T) be two random vectors. Suppose that

$$X \leq_{\rm st} S \tag{1.1}$$

and that

$$[Y\big|X = x] \leq_{\rm st} [T\big|S = s] \quad \text{whenever } x \leq s. \tag{1.2}$$

Then (X, Y) is said to be smaller than (S, T) in the strong stochastic order, and it is denoted as $(X, Y) \leq_{\rm sst} (S, T)$. The multivariate order $\leq_{\rm sst}$ is stronger than the multivariate order $\leq_{\rm st}$.

2. One-sample comparisons

For the proof of the first result we will need the following lemma. To begin with, we recall the definition of the positive dependence concept of *stochastic increasingness*: let (V, W) be a random vector. We say that W is stochastically increasing in V if the conditional random variable $[W\big|V = v]$ is stochastically increasing in $v \in \text{support}\{V\}$ with respect to the univariate ordinary stochastic order $\leq_{\rm st}$.

Lemma 2.1. *Let $(V_1, W_1), (V_2, W_2), \ldots, (V_n, W_n)$ be random vectors. If*

$$V_r \leq_{st} V_{r+1}, \quad r = 1, 2, \ldots, n-1, \tag{2.1}$$

and if

$$W_r \textit{ is stochastically increasing in } V_r \textit{ with respect to } \leq_{st}, \quad r = 1, 2, \ldots, n, \tag{2.2}$$

then

$$(V_r, W_r) \leq_{st} (V_{r+1}, W_{r+1}), \quad r = 1, 2, \ldots, n-1.$$

Proof. From (2.1) it follows that we can construct, on some probability space, random variables $\widehat{V}_1, \widehat{V}_2, \ldots, \widehat{V}_n$ such that $\widehat{V}_r =_{\rm st} V_r$, $r = 1, 2, \ldots, n$, and such that

$$\widehat{V}_1 \leq \widehat{V}_2 \leq \cdots \leq \widehat{V}_n \quad \text{almost surely.} \tag{2.3}$$