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Objective Bayesian analysis for autoregressive models with nugget effects

Cuirong Ren [a,*], Dongchu Sun [b]

[a] Statistical Programs, USPS, Washington DC 20260, USA

[b] Department of Statistics, University of Missouri-Columbia, Columbia, MO 65211, USA

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ABSTRACT

The conditional autoregressive (CAR) and simultaneous autoregressive (SAR) models both have been used extensively for the analysis of spatial structure underlying lattice data in many areas, such as epidemiology, demographics, economics, and geography. Default Bayesian analyses have been conducted recently, but the Bayesian approach has not used or explored these two models with nugget effects. In this paper, we consider general autoregressive models including both CAR and SAR models. The Jeffreys-rule, independence Jeffreys, commonly used reference and "exact" reference priors are derived. The propriety of the marginal priors and joint posteriors is studied for a large class of objective priors. Various Jeffreys and reference priors are shown to yield improper posteriors and only the Jeffreys-rule and the "exact" reference priors yield proper posteriors. We make comparisons for these two objective priors using the frequentist coverage probabilities of the credible intervals. An illustration is given using a real spatial data-set.

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1. Introduction

The conditionally autoregressive (CAR) and simultaneously autoregressive (SAR) models are both very popular models used to model the spatial structure underlying lattice data. CAR models were introduced by Besag [2], while SAR models were originally developed as models on the doubly infinite regular lattice beginning with Whittle [21]. Since then, these models have been used to analyze data in diverse areas, such as epidemiology, demographics, economics and geography. Especially, in the last two decades, because of the convenient employment of Gibbs sampling and more general Markov Chain Monte Carlo (MCMC) methods for fitting certain classes of hierarchical spatial models, the study of both CAR and SAR models is resurgent.

The default (automatic) Bayesian analyses for CAR models were considered, perhaps for the first time, by De Oliveira [9], who studied two versions of Jeffreys priors: the Jeffreys-rule and the independence Jeffreys priors, and made comparisons between these two Jeffreys priors, the maximum likelihood estimates, and the uniform prior. He found that the frequentist properties of Bayesian inferences based on the independence Jeffreys prior were similar or better than other priors. Thus, he recommended the independence Jeffreys prior as the default objective prior. However, in the same paper, the results show that the independence Jeffreys prior does not always yield a proper posterior, so he pointed out that the posterior impropriety is a potential problem when the independence Jeffreys prior is used, although it seems unlikely to be encountered in practice. Consequently, Ren and Sun [16] considered the reference priors for the CAR models and concluded

* Corresponding author.
E-mail address: cuirongren@yahoo.com (C. Ren).

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that only the Jeffreys-rule and the "exact" reference priors can always yield proper posteriors. They finally recommended two "exact" reference priors as the default prior based on the simulation study. This overcomes the difficulty of improper posterior with the independence Jeffreys and commonly used reference priors.

De Oliveira and Song [12] studied objective Bayesian analyses for SAR models. They also considered two versions of the Jeffreys prior. From the simulation results, they also recommended the independence Jeffreys prior as the default prior. They also identified the same problem in this case, that is, the independence Jeffreys prior does not always yield a proper posterior. Ren [15] proposed the reference priors for SAR models and found the reference priors perform better than the Jeffreys-rule prior, so he finally recommended the reference priors as the default priors.

However, the above models studied do not include the measurement error. As we know, the measurement error/ microscale variation is present in many spatial data. In this paper, we will propose objective priors for autoregressive models with the nugget effect, treating both CAR and SAR models as special cases. Although the hierarchical linear mixed models considered by Sun et al. [19] included CAR and SAR models with the nugget effect, they only considered a class of improper noninformative priors and investigated propriety of posteriors under this class of priors. In addition, the class of noninformative priors they considered does not include the objective priors introduced in this paper.

The objective Bayesian approach for spatial data has been studied by many authors. See, for example, [11,14,10,17]. For models with nugget effects, a new parameter was usually introduced in order to conveniently obtain properties and the propriety of posteriors. Ren et al. [17], for example, introduced the noise-to-signal ratio as the new parameter in their paper. For the models we consider in this paper, we also introduce a new parameter, but interestingly, the new parameter is the signal-to-noise ratio instead.

The main object of this paper is to propose default Bayesian analyses for autoregressive models with nugget effects. Jeffreys priors, including Jeffreys-rule and independence Jeffreys priors, the reference priors obtained from asymptotic marginalization or the exact marginalization algorithm, are derived for the parameters in the models and conclusions on propriety of the resulting posteriors for the model parameters are established. Frequentist coverage of the credible intervals for numerical comparisons is described and examples are given based on simulated data to make comparisons between Jeffreys-rule and exact reference priors.

The organization of the paper is as follows. In Section 2, autoregressive models with nugget effects are introduced and the Jeffreys priors and various reference priors are derived. In Section 3, a class of objective priors are introduced, the behavior of integrated likelihood is investigated, and the proprieties of marginal priors and posteriors under all studied objective priors are studied. The two kinds of objective priors, the Jeffreys-rule and "exact" reference priors that always yield the proper posteriors, are compared numerically in terms of frequentist coverage of Bayesian credible intervals, and a real example is used for illustration in Section 4. Section 5 gives a brief summary and comments. Finally, some technical details are given in the Appendix. Several lemmas are present in Appendix A; and the proofs of some main results are present in Appendix B.

2. Autoregressive models and objective priors

2.1. Autoregressive models with nugget effects

We consider a Gaussian Markov random field, where the study area is partitioned into n regions, indexed by integers $1, 2, \ldots, n$. For region i, the variable of interest, y_i, is observed through the following model,

$$y_i = \boldsymbol{x}_i\boldsymbol{\beta} + z_i + e_i, \tag{1}$$

where $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$ is pre-specified, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)' \in \mathbb{R}^p$ are unknown regression parameters, e_i are iid normal $(0, \delta_0)$ errors, and $(z_1, \ldots, z_n)'$ are spatially correlated random effects satisfying

$$(z_i \mid z_j, j \neq i) \sim N\left(\sum_{j=1}^{n} W_{ij}(\rho) z_j, \delta_1\right), \tag{2}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)' \in \mathbb{R}^p$ are unknown regression parameters, $\delta_1 > 0$ and $W_{ij}(\rho)$ are covariance parameters, with $W_{ii}(\rho) = 0$ for all i. It is from [2] that the joint distribution of $\boldsymbol{z}$ is uniquely determined by the full conditional distributions (2). It is often that $W_{ij}(\rho)$ is a linear function of the weight C_{ij}. That is,

$$W_{ij}(\rho) = \rho C_{ij}, \quad \text{for all } i, j. \tag{3}$$

The matrix $\boldsymbol{C} = (C_{ij})_{n\times n}$ is often called a *Weight Matrix* or *Proximity Matrix*.

Often the proximity matrix $\boldsymbol{C}$ is symmetric and treated as known. Common choices of $\boldsymbol{C}$ are as follows.

- *Adjacency Matrix:* $C_{ij} = 1$ if region i and region j share common boundary.
- *k-neighbor Adjacency Matrix:* $C_{ij} = 1$ if region j is one of the k nearest neighbors of region i.
- *Distance Matrix:* $C_{ij} =$ the distance between centroids of regions i and j.

The case of *Adjacency* matrix was introduced in [3] and the other two cases can be found in [4,18].

Note that (2) and the joint distribution based on (3) are equivalent to

$$\boldsymbol{z} \sim N_n(\boldsymbol{0}, \delta_1(\boldsymbol{I}_n - \rho\boldsymbol{C})^{-1}). \tag{4}$$