Journal of Multivariate Analysis 130 (2014) 1–20

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

A note on the computation of sharp numerical bounds for the distribution of the sum, product or ratio of dependent risks

Hélène Cossette[a], Marie-Pier Côté[b], Mélina Mailhot[c,*], Etienne Marceau[a]

[a] École d'Actuariat, Université Laval, Canada
[b] Department of Mathematics and Statistics, McGill University, Canada
[c] Department of Mathematics and Statistics, Concordia University, Canada

ARTICLE INFO

Article history:
Received 22 August 2013
Available online 13 May 2014

AMS 2010 subject classifications:
62H00
62P05

Keywords:
Risk aggregation
Dependent random variables
Numerical bounds
Value-at-Risk

ABSTRACT

In this paper, an approximation method for computing numerically the cumulative distribution function of the sum of d random variables is developed. The method leads to numerical bounds for the distribution of the sum of dependent risks. The bounds are fast to compute and converge to the exact value if the joint probability density function exists. They also allow to evaluate sharp numerical bounds on the Value-at-Risk measure. Moreover, the fact that the approximation is deterministic, hence without uncertainty on the resulting values, is an advantage over MC simulation techniques. Applications in actuarial science and finance illustrate the accuracy of the procedure. We also present analogous bounds for the distribution of the product or the ratio of two random variables, which can be useful for actuarial or financial applications.

1. Introduction

Situations where the distribution of a sum of dependent risks needs to be computed are frequent in actuarial science, in quantitative risk management, in statistics and in applied probability in general. Calculation of overall capital charge for a portfolio of risks, evaluation and analysis of risk measures for decision making, and strategic planning are circumstances where knowledge of the cumulative distribution function (cdf) of dependent random variables (rvs) is crucial. Assuming independence between risks allows a fast computation of their aggregate distribution, but may be inappropriate and induce wrong results. Many quantities of interest, such as risk measures, are computed using the distribution of aggregated risks.

Several techniques have been developed to approximate or to calculate bounds if there is no closed-form expression for the cdf of the sum of dependent rvs. [3] proposes a numerical method for the evaluation of the cdf of a sum of rvs. Their approach is based on the discretization of the marginals (see e.g. [12]) and leads to lower and upper bounds on the exact value. [1] developed the AEP algorithm for fast computation of the distribution of the aggregated risks. Their numerical method is deterministic and provides accurate point approximations for the cdf of a sum of rvs. [5] suggests methods to approximate the distribution of dependent risks, based on compound Poisson distributions.

In this paper, we introduce a method inspired by the AEP algorithm. We develop lower and upper numerical bounds for the cdf of a sum of rvs that converge to the exact value when the joint probability density function (pdf) exists. The

* Corresponding author.
E-mail address: melina.mailhot@concordia.ca (M. Mailhot).

http://dx.doi.org/10.1016/j.jmva.2014.04.023

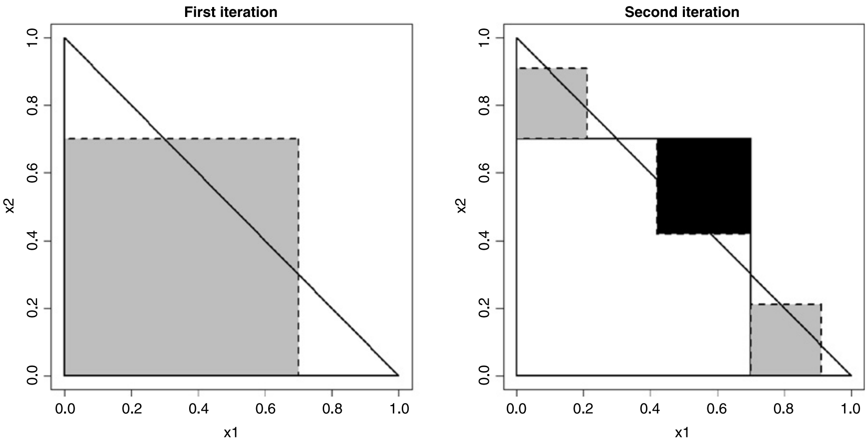

Fig. 1. Illustration of the AEP algorithm in two dimensions, for $s = 1$ and $\alpha_{AEP} = 0.7$.

numerical bounds are fast to compute and have many advantages in comparison to simulation techniques. It allows the computation of two numerical bounds for the Value-at-Risk (VaR) of the sum of dependent risks. Our approach can also be seen as a complement to the numerical approach proposed by [1].

Products and ratios of rvs are also interesting in actuarial and financial applications. A product of two rvs X_1 and X_2 could arise in evaluating the accumulated value of an investment, or the payment from a reinsurance agreement (if X_1 is the total loss and X_2 is for example an indicator of some event, or a percentage based on an index). In an insurance context, it is of interest to model the loss ratio, defined as the total loss divided by the income over the period. Both components can be dependent, especially due to reinsurance agreements and/or return on assets. This motivates the extension of our method to evaluate bounds on the cdf of product or ratio of rvs.

The outline of the paper is as follows. In Section 2, we present the algorithm in the bivariate and trivariate cases, and then we show expressions for the general case, along with the proof of convergence. In Section 2.5, numerical bounds on the risk measure VaR are developed. Section 3 presents many applications and comparison of our approach with the one of [1]. Finally, the last two sections extend the method to the cdf of the product or the ratio of two rvs, respectively.

2. Description of the method

This section aims to describe the algorithm proposed and shows its easiness of implementation, in two and three dimensions. We first briefly explain the AEP algorithm and our motivation for developing our method. Then, we present the bivariate and trivariate cases. The results are extended to higher dimensions in Section 2.4.

2.1. AEP algorithm and motivation

Let us start by describing the AEP method from [1]. The main idea of the AEP algorithm is to calculate the cdf of the sum of d rvs, evaluated at a constant s. This is equivalent to evaluating the probability that the random vector $(X_1, \dots, X_d)$ takes values in a d-simplex, which is represented by the region in the space of $X_1, \dots, X_d$ such that $X_1 + \cdots + X_d \leq s$. In the first iteration, we calculate the probability in a hypercube of the same dimension as the number of aggregated random variables, with edge length equal to $s\alpha_{AEP}$, and $\alpha_{AEP} \in [1/d, 1)$. This first step creates new smaller d-simplexes around the hypercube, so new hypercubes are added and removed, until the initial simplex is almost totally filled. The reader is referred to [1] for the detailed procedure. Throughout this paper, the rvs $X_1, \dots, X_d$ are assumed to be non-negative as is often the case in actuarial applications. However, [1] also discussed the extension to random vector bounded from below.

For instance, in two dimensions, Fig. 1 illustrates the steps followed by the AEP algorithm, after one and two iterations (left and right figure respectively), for $s = 1$ and $\alpha_{AEP} = 0.7$. On these figures, the probabilities in the gray squares are added, while the probability in the black square is subtracted. At the end of the second iteration, nine new triangles are formed and they will be approximated by squares in the third iteration, and so on. As the number of iterations increases, the new triangles are smaller and the accuracy of the approximation is improved. This algorithm converges quickly for clever values of α_{AEP}; [1] provides optimal values for this parameter (e.g. $\frac{2}{3}$ for $d = 2$ and $\frac{1}{2}$ for $d = 3$).

From Fig. 1, one can see that the approximation may either underestimate or overestimate the true value of the cdf when $\alpha_{AEP} > 1/d$. In order to compute a conservative value of some risk measures in insurance or finance applications, it is of interest to understand if the approximation lies below or above the true value. For $\alpha_{AEP} = 0.5$ in two dimensions, the approximation will always lie below the true value, as illustrated in Fig. 2. This corresponds to our lower bound in two dimensions, when we reformulate the problem in terms of rectangles. We also propose an upper bound based on rectangles