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Dependence properties of multivariate max-stable distributions

Ioannis Papastathopoulos [a,*], Jonathan A. Tawn [b]

[a] *School of Mathematics, University of Bristol, United Kingdom*

[b] *Department of Mathematics and Statistics, Lancaster University, United Kingdom*

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ABSTRACT

For an m-dimensional multivariate extreme value distribution there exist $2^m - 1$ exponent measures which are linked and completely characterise the dependence of the distribution and all of its lower dimensional margins. In this paper we generalise the inequalities of Schlather and Tawn (2002) for the sets of extremal coefficients and construct bounds that higher order exponent measures need to satisfy to be consistent with lower order exponent measures. Subsequently we construct nonparametric estimators of the exponent measures which impose, through a likelihood-based procedure, the new dependence constraints and provide an improvement on the unconstrained estimators.

1. Introduction

Max-stable distributions arise naturally from the study of limiting distributions of appropriately scaled componentwise maxima of independent and identically distributed random variables. Here and throughout the vector algebra is to be interpreted as componentwise. A random vector $Y = (Y_1, \dots, Y_m)$ with unit Fréchet margins, i.e., $G_i(y) := \mathbb{P}(Y_i < y) = \exp(-1/y), y > 0, i \in M_m = \{1, \dots, m\}$, is called max-stable if its distribution function is max-stable, i.e., if

$$G_{M_m}(y_{M_m}) := \mathbb{P}\left(Y < y_{M_m}\right) = \exp\left\{-\int_{S_m} \max_{i \in M_m}\left(\frac{w_i}{y_i}\right) dH(w_1, \dots, w_m)\right\}, \tag{1}$$

where $y_{M_m} = (y_1, \dots, y_m) \in \mathbb{R}_+^m$, $S_m = \left\{(w_1, \dots, w_m) \in \mathbb{R}_+^m : \sum_{i=1}^m w_i = 1\right\}$ is the $(m-1)$-dimensional unit simplex and H is an arbitrary finite measure, with total mass m, that satisfies

$$\int_{S_m} w_i dH(w_1, \dots, w_m) = 1 \quad \text{for any } i \in M_m.$$

The last condition is necessary for G_{M_m} to have unit Fréchet margins and representation (1) is due to Pickands [8]. There is no loss of generality in assuming unit Fréchet margins since our focus is placed on the dependence structure of max-stable distributions, i.e., we are interested in the copula function [7] which is invariant to strictly monotone marginal transformations and in practice we can standardise random variables to unit Fréchet margins.

* Corresponding author.
E-mail addresses: i.papastathopoulos@bristol.ac.uk (I. Papastathopoulos), j.tawn@lancaster.ac.uk (J.A. Tawn).

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The dependence properties of max-stable distributions have received attention in the multivariate extreme value literature. Dating back to Sibuya [12] and Tiago de Oliveira [14], it has been known that max-stable distributions are necessarily positively quadrant dependent, i.e.,

$$G_{M_m}(y_{M_m}) \geq \prod_{i=1}^{m} G_i(y_i) \quad y_{M_m} \in \mathbb{R}_+^m, \tag{2}$$

which implies that no pair of random variables can be negatively dependent. Additionally, max-stable distributions satisfy even stronger forms of dependence. Marshall and Olkin [6] show that $\mathrm{Cov}\{g(Y), h(Y)\} \geq 0$ for every pair of non-decreasing real functions g and h on $\mathbb{R}^m$, i.e., they are associated. For a review of the dependence properties of max-stable distributions we refer the reader to [1] and the references therein.

Although all of the aforementioned properties exhibit characteristics for the dependence structure of the class of max-stable distributions, they are far too general to be either tested or implemented in practice. In this paper, we introduce additional inequalities for the dependence structure that can be incorporated, through a likelihood-based procedure, into the estimation of max-stable distributions from observed componentwise maxima. In particular, the inequalities yield a set of natural constraints for the parameter space of multivariate extreme value distributions and constrained maximum likelihood optimisation subject to the additional constraints is proposed. This leads to new more efficient non-parametric estimators that satisfy the additional constraints and can be used in practice to improve inference.

The new constraints are in essence the generalisation of the Schlather and Tawn [10,11] inequalities for the extremal coefficients which correspond to the dependence properties of max-stable distributions for the special case of $G_{M_m}(y, \ldots, y)$, $y > 0$. As such, our notation and strategy are influenced by the work of Schlather and Tawn [10,11]. The new inequalities presented in this paper are related to the general case of $G_{M_m}(y_{M_m})$, $y_{M_m} \in \mathbb{R}_+^m$.

In Section 2 we introduce the class of max-stable distributions along with the Schlather and Tawn [10] inequalities for the extremal coefficients. Subsequently, we present the general result of the paper that gives rise to inequalities for the exponent measures. In Section 3 we consider the Hall and Tajvidi [4] nonparametric estimator for the exponent measure and extend it, through a likelihood-based procedure, to satisfy the new inequalities. Finally, in Section 4 a simulation study is conducted to assess the performance of the constrained estimator.

2. Dependence properties

2.1. Background

The class of max-stable distributions arises naturally from the study of appropriately scaled component-wise maxima of random variables. Consider a set of independent and identically distributed random vectors $X^j = (X_1^j, \ldots, X_m^j)$, $j = 1, \ldots, n$, with unit Fréchet margins. Under weak conditions [9] it follows that

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1,\ldots,m} \left\{\max_{j=1,\ldots,n} X_i^j/n < y_i\right\}\right) = G_{M_m}(y_{M_m}), \quad y_{M_m} \in \mathbb{R}_+^m. \tag{3}$$

The distribution function G_{M_m} can be completely characterised by the following representations:

$$\begin{aligned} V_{M_m}(y_{M_m}) = -\log G_{M_m}(y) &= \int_{S_m} \max_{i\in M_m}\left(\frac{w_i}{y_i}\right) dH(w_1, \ldots, w_m), \\ &= \left\{\sum_{i=1}^{m} 1/y_i\right\} A_{M_m}\left(\frac{1/y_1}{\sum_{i=1}^{m} 1/y_i}, \ldots, \frac{1/y_m}{\sum_{i=1}^{m} 1/y_i}\right), \end{aligned} \tag{4}$$

where the function V_{M_m} is known as the exponent measure of the multivariate extreme value distribution G_{M_m} and A_{M_m}, called the *Pickands' dependence function*, is a convex function that satisfies $\max\{w_1, \ldots, w_m\} \leq A_{M_m}(w_1, \ldots, w_m) \leq 1$, $(w_1, \ldots, w_m) \in S_m$. This condition implies that $A_{M_m}(e_j) = 1, j \in M_m$, where e_j is the jth unit vector in $\mathbb{R}^m$.

Let $C_m = 2^{M_m} \setminus \{\emptyset\}$ and denote also by $y_B = \{y_i : i \in B\}$ for $B \in C_m$. Then we can define $2^m - 1$ exponent measures for an m-dimensional max-stable random vector Y, where each one characterises completely the distribution function of a marginal random variable Y_B of Y, i.e.,

$$V_B(y_B) = -\log\{\mathbb{P}(Y_B < y_B)\} = -\log\left\{\lim_{y_{M_m\setminus B}\to\infty} G_{M_m}(y_{M_m})\right\}, \quad B \in C_m.$$

The set of exponent measures $\{V_B : B \in C_m\}$ describes completely the dependence structure of a max-stable distribution given by Eq. (1) and all of its lower dimensional margins. Also, with each exponent measure V_B there is an