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Infinitely divisible multivariate and matrix Gamma distributions

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ABSTRACT

Classes of multivariate and cone valued infinitely divisible Gamma distributions are introduced. Particular emphasis is put on the cone-valued case, due to the relevance of infinitely divisible distributions on the positive semi-definite matrices in applications. The cone-valued class of generalised Gamma convolutions is studied. In particular, a characterisation in terms of an Itô–Wiener integral with respect to an infinitely divisible random measure associated to the jumps of a Lévy process is established.

A new example of an infinitely divisible positive definite Gamma random matrix is introduced. It has properties which make it appealing for modelling under an infinite divisibility framework. An interesting relation of the moments of the Lévy measure and the Wishart distribution is highlighted which we suppose to be important when considering the limiting distribution of the eigenvalues.

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1. Introduction

The classical examples of multivariate and matrix Gamma distributions in the probability and statistics literature are not necessarily infinitely divisible [14,19,40]. These examples are analogous to one-dimensional Gamma distributions and are obtained by a direct generalisation of the one-dimensional probability densities; see for example [15,23,24]. Working in the domain of Fourier transforms, some infinitely divisible matrix Gamma distributions have recently been considered in [5,27]. Their Lévy measures are direct generalisations of the one-dimensional Gamma distribution. The work of [27] arose in the context of random matrix models relating classical and free infinitely divisible distributions.

The study of infinitely divisible random elements in cones has been considered in [4,25,26,31] and references therein. They are important in the construction and modelling of cone increasing Lévy processes. In the particular case of infinitely divisible positive-definite random matrices, their importance in applications has been recently highlighted in [7,8,28,29]. This is due to the fact that infinite divisibility allows modelling by matrix Lévy and Ornstein–Uhlenbeck processes, which are in those papers used to model the time dynamics of a $d \times d$ covariance matrix to obtain a so-called stochastic volatility model (for observed series of financial data).

Generalised Gamma Convolutions (GGC) is a rich and interesting class of one-dimensional infinitely divisible distributions on the cone $\mathbb{R}_+ = [0, \infty)$. It is the smallest class of infinitely divisible distributions on $\mathbb{R}_+$ that contains all Gamma

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distributions and that is closed under classical convolution and weak convergence. This class was introduced by O. Thorin in a series of papers and further studied by L. Bondesson in his book [10]. The book of Steutel and Van Harn [39] contains also many results and examples about GGC. Several well known and important distributions on $\mathbb{R}_+$ are GGC. The recent survey paper by James, Roynette and Yor [16] contains a number of classical results and old and new examples of GGC. The multivariate case was considered in Barndorff-Nielsen, Maejima and Sato [3].

There are three main purposes in this paper. We formulate and study multivariate and cone valued Gamma distributions which are infinitely divisible. Second, we consider and characterise the corresponding class $GGC(K)$ of Generalised Gamma Convolutions on a finite dimensional cone K. Finally, we introduce a new example of a positive definite random matrix with infinitely divisible Gamma distribution and with explicit Lévy measure.

The main results and organisation of the paper are as follows. Section 2 briefly presents preliminaries on notation and results about one-dimensional GGC on $\mathbb{R}_+$ as well as some matrix notation. Section 3 introduces a class of infinitely divisible d-variate Gamma distributions $\Gamma_d(\alpha, \beta)$, whose Lévy measures are analogous to the Lévy measure of the one-dimensional Gamma distribution. The parameters α and β are measures and functions on $\mathbf{S}$ (the unit sphere with respect to a prescribed norm), respectively. It is shown that the distribution does not depend on the particular norm under consideration. The characteristic function is derived and it is shown that the Fourier–Laplace transform on $\mathbb{C}^d$ exists if β is bounded away from zero α- almost everywhere. Furthermore, the finiteness of moments of all orders is studied and some interesting examples exhibiting essential differences to univariate Gamma distributions are given.

Section 4 considers cone valued Gamma distributions and their corresponding class $GGC(K)$ of Generalised Gamma Convolutions on a cone K, defined as the smallest class of distributions on K which is closed under convolution and weak convergence and contains all the so-called elementary Gamma variables in K (and also all Gamma random variables in K in our new definition). This class is characterised as the stochastic integral of a non-random function with respect to the Poisson random measure of the jumps of a Gamma Lévy process on the cone. This is a new representation in the multivariate case extending the Wiener–Gamma integral characterisation of one-dimensional GGC on $\mathbb{R}_+ = [0, \infty)$, as considered, for example, in [16].

Section 5 considers the special cone valued case of infinitely divisible positive-semidefinite $d \times d$ matrix Gamma distributions. New examples are introduced via an explicit form of their Lévy measure. They include as particular cases the examples considered in [5,27]. A detailed study is done of the new two parameter positive definite matrix distribution $A\Gamma(\eta, \Sigma)$, where $\eta > (d-1)/2$ and Σ is a $d \times d$ positive definite matrix. This special infinitely divisible Gamma matrix distribution has several modelling features similar to the classical (but non-infinitely divisible) matrix Gamma distribution defined through a density, in particular the Wishart distribution. Namely, moments of all orders exist, the matrix mean is proportional to Σ and the matrix of covariances equals the second moment of the Wishart distribution. When Σ is the $d \times d$ identity matrix I_d, the distribution is invariant under orthogonal conjugations and the trace of a random matrix M with distribution $A\Gamma(\eta, \mathrm{I}_d)$ has a one-dimensional Gamma distribution. A relation of the moments of the Marchenko–Pastur distribution with the asymptotic moments of the Lévy measure is exhibited. Hence, this matrix Gamma distribution has a special role when dealing with a random covariance matrix and its time dynamics, e.g. by specifying it as a matrix Lévy or Ornstein–Uhlenbeck process. As an application, the matrix normal-Gamma distribution is introduced, which is a matrix extension of the one-dimensional variance Gamma distribution of [22] which is popular in finance.

2. Preliminaries

For the general background in infinitely divisible distributions and Lévy processes we refer to the standard references, e.g. [36].

2.1. One-dimensional GGC

A positive random variable Y with law $\mu = \mathscr{L}(Y)$ belongs to the class of Generalised Gamma Convolutions (GGC) on $\mathbb{R}_+ = [0, \infty)$, denoted by $T(\mathbb{R}_+)$, if and only if there exists a positive Radon measure υ_μ on $(0, \infty)$ and $a > 0$ such that its Laplace transform is given by:

$$L_\mu(z) = \mathbb{E}e^{-zY} = \exp\left(-az - \int_0^\infty \ln\left(1 + \frac{z}{s}\right) \upsilon_\mu(\mathrm{d}s)\right) \tag{2.1}$$

with

$$\int_0^1 |\log x| \upsilon_\mu(\mathrm{d}x) < \infty, \qquad \int_1^\infty \frac{\upsilon_\mu(\mathrm{d}x)}{x} < \infty. \tag{2.2}$$

For convenience we shall work without the translation term, *i.e.* with $a = 0$. The measure υ_μ is called the Thorin measure of μ. Its Lévy measure is concentrated on $(0, \infty)$ and is such that:

$$\nu_\mu(\mathrm{d}x) = x^{-1} l_\mu(x) \mathrm{d}x, \tag{2.3}$$

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