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Tests for real and complex unit roots in vector autoregressive models

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ABSTRACT

The article proposes new tests for the number of real and complex unit roots in vector autoregressive models. The tests are based on the eigenvalues of the sample companion matrix. The limiting distributions of the eigenvalues converging to the unit eigenvalues turn out to be of a non-standard form and expressible in terms of Brownian motions. The tests are defined such that the null distributions related to eigenvalues ± 1 are the same. The tests for the unit eigenvalues with nonzero imaginary part are defined independently of the angular frequency. When the tests are adjusted for deterministic terms, the null distributions usually change. Critical values are tabulated via simulations. Also some simulation based finite sample properties are presented together with comparisons with corresponding likelihood ratio tests. The relation of the unit roots to cointegration is discussed. An empirical example is provided to show how to use the test with real data.

1. Introduction

The vector autoregressive model has been widely used especially in econometrics. Special interest is devoted to separating out common trends and stationary components. There is a large literature on these issues under the title of cointegration analysis. The authoritative monograph is [10]. Hylleberg, Engle, Granger, and Yoo [9] introduced the concept of cointegration to seasonal models. Further analysis is developed by Ahn and Reinsel [3], Johansen and Schaumburg [11], Cubadda [7] and Ahn, Cho, and Seong [2] among others. These works are based on the Gaussian likelihood. Another approach is based on the eigenvalues of the sample companion matrix estimated by least squares, see [1] and references therein. Here we extend the eigenvalue approach to complex unit roots, the seasonal unit roots being the most important special cases. We derive the asymptotic distributions of the sample eigenvalues of the companion matrix under the assumption that there are given number of real and complex unit roots with the rest being less than one in absolute value. We also allow deterministic terms such as constant level and different types of trends. It turns out that the distributions depend on the angular frequency θ of the complex unit roots $e^{\pm i\theta}$ as well as the type of deterministic terms present. The proposed test statistics, however, are such that their limiting distributions reduce to two types: those associated with eigenvalues ± 1 and those associated with the complex conjugate pair. In the latter case limiting distributions are independent of the angular frequency. This fact is useful because the test can be applied without knowing the exact value of the angular frequency.

The article is organized as follows. In Section 2 the limiting distributions are derived at different angular frequencies with and without deterministic terms. Section 3 shows how the unit eigenvalues are related to cointegration. Section 4

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defines the proposed tests for the number of unit roots and tabulates necessary critical values. Simulation results on the finite sample behavior of the tests are presented in Section 5. Tests are applied to income and consumption data from West Germany in Section 6. Results are proved in Section 7. All numerical work is done within R environment [15].

2. Limiting distributions

Consider the p-variate vector autoregressive model of order d, VAR(d),

$$\boldsymbol{y}_t = \boldsymbol{\Pi}_1 \boldsymbol{y}_{t-1} + \cdots + \boldsymbol{\Pi}_d \boldsymbol{y}_{t-d} + \boldsymbol{\epsilon}_t, \quad t = 1, \ldots, T, \tag{2.1}$$

where the errors $\boldsymbol{\epsilon}_t$ are independent with $E(\boldsymbol{\epsilon}_t) = 0$, and with cov $(\boldsymbol{\epsilon}_t) = \boldsymbol{\Omega}$ assumed positive definite (p.d.). Let us assume that the initial values $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{-d+1}$ are fixed. Write the VAR(d) model (2.1) in companion form

$$\boldsymbol{Y}_t = \boldsymbol{\Psi} \boldsymbol{Y}_{t-1} + \boldsymbol{\eta}_t, \tag{2.2}$$

where

$$\boldsymbol{Y}_t = \begin{bmatrix} \boldsymbol{y}_t \\ \vdots \\ \boldsymbol{y}_{t-d+1} \end{bmatrix}, \quad \boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Pi}_1 & \boldsymbol{\Pi}_2 & \cdots & \boldsymbol{\Pi}_{d-1} & \boldsymbol{\Pi}_d \\ \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{\eta}_t = \begin{bmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}. \tag{2.3}$$

In this article we derive statistical tests concerning unit eigenvalues, real and complex, of $\boldsymbol{\Psi}$. The eigenvalues are the roots of the determinant equation $|\boldsymbol{\Psi} - \lambda \boldsymbol{I}| = 0$. All unit roots can be written in the form $e^{i\theta}$. We assume that there are q_0 roots equal to $e^{i0} = 1$, q_j pairs of roots equal to $e^{\pm i\theta_j}$, $0 < \theta_j < \pi$, $j = 1, \ldots, n$, and q_{n+1} roots equal to $e^{i\pi} = -1$. Additionally there are s eigenvalues inside the unit disc. Finally, we assume that there are no eigenvalues outside the unit disc. Then $q_0 + 2(q_1 + \cdots + q_n) + q_{n+1} + s = pd$. The numbers q_j, are called algebraic multiplicities, but we also assume that there are q_j linearly independent eigenvectors corresponding to each unit eigenvalue, i.e. geometric multiplicity equals to the algebraic multiplicity for the unit roots.

The matrix $\boldsymbol{\Psi}$ is estimated by the least squares (LS)

$$\widehat{\boldsymbol{\Psi}} = \left(\sum_{t=1}^{T} \boldsymbol{Y}_t \boldsymbol{Y}_{t-1}' \right) \left(\sum_{t=1}^{T} \boldsymbol{Y}_{t-1} \boldsymbol{Y}_{t-1}' \right)^{-1}, \tag{2.4}$$

where the prime refers to transpose. Note that $\widehat{\boldsymbol{\Psi}}$ has the same structure as $\boldsymbol{\Psi}$ in (2.3). The eigenvalues of the estimate $\widehat{\boldsymbol{\Psi}}$ are exploited in our tests determining the number of unit roots corresponding to each frequency θ_j, $j = 0, 1, \ldots, n+1$. The limiting distributions of the eigenvalues of $\widehat{\boldsymbol{\Psi}}$ corresponding to the unit eigenvalues of $\boldsymbol{\Psi}$ are given in the next theorem which is proved in Section 7.

In the following we need complex matrices and vectors. If $\boldsymbol{A}$ is complex, then its complex conjugate is $\bar{\boldsymbol{A}}$ and the transpose of the complex conjugate is $\boldsymbol{A}^* = \bar{\boldsymbol{A}}'$.

Theorem 1. *Let $\widehat{\boldsymbol{\Psi}}$ be the least squares estimate of $\boldsymbol{\Psi}$ in model (2.2). Assume that $\boldsymbol{\Psi}$ has q_0 eigenvalues equal to 1, q_j pairs eigenvalues equal to $e^{\pm i\theta_j}$, $0 < \theta_j < \pi$, $j = 1, \ldots, n$, q_{n+1} eigenvalues equal to -1. Further we assume that the multiplicities $q_0, \ldots, q_{n+1}$ are equal to geometric multiplicities. All other eigenvalues are assumed to lie inside the unit disc on the complex plane. All limits are taken as $T \to \infty$.*

(1) *If $q_0 > 0$, let $\hat{\lambda}_{01}, \ldots, \hat{\lambda}_{0q_0}$ be the q_0 eigenvalues of $\widehat{\boldsymbol{\Psi}}$ which are closest to 1. Then $T(\hat{\lambda}_{01} - 1, \ldots, \hat{\lambda}_{0q_0} - 1)'$ converges in distribution to the eigenvalues of the random matrix*

$$\int_0^1 \mathrm{d}\boldsymbol{Z}(s)\, \boldsymbol{Z}(s)' \left(\int_0^1 \boldsymbol{Z}(s)\boldsymbol{Z}(s)'\, \mathrm{d}s \right)^{-1}, \tag{2.5}$$

where $\boldsymbol{Z}$ is a q_0-dimensional standard Brownian motion.

(2) *If $q_{n+1} > 0$, let $\hat{\lambda}_{\pi 1}, \ldots, \hat{\lambda}_{\pi q_{n+1}}$ be the q_{n+1} eigenvalues of $\widehat{\boldsymbol{\Psi}}$ closest to -1. Then $-T(\hat{\lambda}_{\pi 1} + 1, \ldots, \hat{\lambda}_{\pi q_{n+1}} + 1)'$ converges in distribution to the eigenvalues of the random matrix (2.5) with $\boldsymbol{Z}$ being now a q_{n+1}-dimensional standard Brownian motion.*

(3) *If $q_j > 0$, let $\hat{\lambda}_{\theta_j 1}, \ldots, \hat{\lambda}_{\theta_j q_j}$ be the q_j eigenvalues of $\widehat{\boldsymbol{\Psi}}$ which are closest to $e^{i\theta_j}$. Then $T(\hat{\lambda}_{\theta_j 1} - e^{i\theta_j}, \ldots, \hat{\lambda}_{\theta_j q_j} - e^{i\theta_j})'$ converges in distribution to the eigenvalues of the random matrix*

$$e^{i\theta_j} \int_0^1 \mathrm{d}\boldsymbol{\Lambda}_j(s)\, \boldsymbol{\Lambda}_j(s)^* \left(\int_0^1 \boldsymbol{\Lambda}_j(s)\boldsymbol{\Lambda}_j(s)^*\, \mathrm{d}s \right)^{-1}, \tag{2.6}$$

and $T(\bar{\hat{\lambda}}_{\theta_j 1} - e^{-i\theta_j}, \ldots, \bar{\hat{\lambda}}_{\theta_j q_j} - e^{-i\theta_j})'$ converges in distribution to the eigenvalues of the complex conjugate of (2.6). Here $\boldsymbol{\Lambda}_j = \boldsymbol{\Lambda}_{1j} + i\boldsymbol{\Lambda}_{2j}$ with $\boldsymbol{\Lambda}_{1j}$ and $\boldsymbol{\Lambda}_{2j}$ being independent q_j-dimensional Brownian motions such that $\boldsymbol{\Lambda}_j$ is the standard complex valued Brownian motion with $E[\boldsymbol{\Lambda}_j(s)\boldsymbol{\Lambda}_j(u)^*] = \min(s, u)\boldsymbol{I}$.