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Characterization of Gaussian distribution on a Hilbert space from samples of random size

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ABSTRACT

We obtain two characterizations of the Gaussian distribution on a Hilbert space from samples of random size.

1. Introduction

Several characterizations of the univariate and the multivariate normal distributions are known (cf. Kagan et al. [3] and Prakasa Rao [9]). Most of these results involve statistics based on fixed sample sizes. However there are situations, such as in the study of population growth using branching processes, the size of a generation depends on the size of the previous generation which itself is random. For the breeding habit and study growth of an organism in one generation, one needs to study distributions of statistics based on population sizes of the previous generation which in turn are random. In such cases, it is necessary to characterize the underlying distribution based on samples of random size. Cook [1] obtained a characterization of correlated normal random vectors. Kagan and Shalaevski [4]) obtained the characterization of normal distribution by a property of the non-central chi-square distribution. Kotlarski and Cook [5] extended the results in Cook [1] and Kagan and Shalaevski [4] and obtained two characterizations of the multivariate normal distribution based on samples of random size. Prakasa Rao [8] obtained similar results in an unpublished report. In view of the recent development of methods of functional data analysis, it would be of interest to investigate whether the results on characterizations of Gaussian distribution obtained in the case of Euclidean spaces R and R^k continue to hold when the observation space is a function space such as a Hilbert space. Our aim is to characterize the Gaussian distribution on a real separable Hilbert space H from samples of random size. Example of such a space H is the space of square integrable functions f on the real line associated with the norm

$$\|f\| = \left[\int_R |f(x)|^2 dx\right]^{1/2}.$$

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2. Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and H be a real separable Hilbert space. Let $\mathcal{B}$ be the Borel-σ-algebra generated by the norm topology on the space H. A mapping $X : \Omega \to H$ is said to be a *random element* taking values in a Hilbert space H if $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$. Define

$$\mu_X(B) = \mu(X^{-1}(B)), \quad B \in \mathcal{B}.$$

It is easy to check that μ_X is a probability measure on the measurable space $(H, \mathcal{B})$. Let $\mathcal{M}(H)$ denote the class of all probability measures on $(H, \mathcal{B})$. Let $\langle x, y\rangle$ denote the inner product and $\|x\|$ the norm defined on the Hilbert space H. Let $\nu \in \mathcal{M}(H)$ be such that

$$\int_H \|x\|^2 \nu(dx) < \infty.$$

Then the *covariance operator* S of ν is the Hermitian operator determined uniquely by the quadratic form

$$\langle Sy, y\rangle = \int_H \langle x, y\rangle^2 \nu(dx).$$

A positive definite Hermitian operator S on the Hilbert space H is called an S-operator if it has finite trace, that is, for some orthonormal basis $\{e_i, i \geq 1\}$, of the Hilbert space H, the sum $\sum_{i=1}^{\infty} \langle Se_i, e_i\rangle < \infty$. In such a case, the inequality holds for every orthonormal basis of the Hilbert space H.

Suppose ν is a probability measure in $\mathcal{M}(H)$ such that

$$\int_H \|x\| \nu(dx) < \infty.$$

It can be shown that there exists an element x_0 in H such that

$$\langle x_0, y\rangle = \int_H \langle x, y\rangle \nu(dx), \quad y \in H.$$

The element x_0 is called the mean of the probability measure ν or the expectation of the random element X if the distribution of the random element X is ν. We denote the mean or expectation x_0 by the notation

$$\int_H x\ \nu(dx).$$

For any probability measure ν on the measurable space $(H, \mathcal{B})$, the *characteristic functional* $\hat{\nu}(.)$ is a functional defined on H by the relation

$$\hat{\nu}(y) = \int_H e^{i\langle x,y\rangle} \nu(dx), \quad y \in H.$$

The characteristic functional $\phi_X(.)$ of the random element X is given by

$$\begin{aligned}\phi_X(y) &= \int_H e^{i\langle x,y\rangle} \mu_X(dx), \quad y \in H \\ &= \int_\Omega e^{i\langle X(\omega),y\rangle} \mu(d\omega), \quad y \in H.\end{aligned}$$

It is known that there is a one-to-one correspondence between the characteristic functionals and the probability measures on H. Furthermore the characteristic functional ϕ_X of a random element X satisfies the conditions

$$\phi_X(0) = 1, \quad |\phi_X(y)| \leq 1, \qquad \phi_X(y) = \overline{\phi_X(-y)}, \quad y \in H$$

where 0 denotes the identity element in H. Moreover the function $\phi_X(.)$ is continuous in the norm topology. In addition, if X and Y are independent random elements taking values in H, then $X + Y$ is also a random element taking values in H, and

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t), \quad t \in H.$$

For proofs of these results, see Parthasarathy [7] or Grenander [2].

A probability measure μ_X generated by a random element X on a Hilbert space H is said to be *Gaussian* if its characteristic functional $\phi_X(y)$ is of the form

$$\phi_X(y) = \exp\left\{i\langle x_0, y\rangle - \frac{1}{2}\langle Sy, y\rangle\right\}$$

where x_0 is a fixed element in H and S is an S-operator on H. It can be shown that x_0 is the mean and the operator S is the covariance operator for the Gaussian measure with characteristic functional $\phi_X(y)$, $y \in H$ (cf. Grenander [2], Theorem 6.3.1.). The following result is due to Grenander [2], p. 141.