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On the convergence of the spectrum of finite order approximations of stationary time series

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HIGHLIGHTS

- Conditions for L_2 convergence of the spectral density of AR and MA approximations.
- Convergence in mean of the spectral density based on covariance estimates.
- Convergence of the spectral density at origin for both cases.

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ABSTRACT

This paper is on the asymptotic behavior of the spectral density of finite autoregressive (AR) and moving average (MA) approximations for a wide sense stationary time series. We consider two aspects: convergence of spectral density of moving average and autoregressive approximations when the covariances are known and when they are estimated. Under certain mild conditions on the spectral density and the covariance sequence, it is shown that the spectral densities of both approximations converge in L_2 as the order of approximation increases. It is also shown that the spectral density of AR approximations converges at the origin under the same conditions. Under additional regularity assumptions, we show that similar results hold for approximations from empirical covariance estimates.

1. Introduction

Linear estimation of a time series from a finite number of past observations is a problem encountered in a diverse variety of applications including econometrics, statistical signal processing and neuroscience. This paper deals with the spectral properties of finite order linear approximations for a *regular* zero-mean, real-valued wide sense stationary (WSS) sequence with finite second moment. From the Wold decomposition [14,37], it follows that such a process can be expressed as an infinite weighted sum of unit variance, uncorrelated random variables called the *innovation process*. This is a moving average

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(MA) type model for the process. By a one-step application of the projection theorem, on the other hand, one obtains an infinite order autoregressive (AR) model of the process, wherein the process is expressed as a weighted sum of all its past values, plus the current value of the innovation process.

The problem of approximating an infinite-order process and its spectral density using a finite order model has had a long history of research [13] and a detailed survey of the key results in this area is available in [27]. A number of problems have been studied related to choosing the optimal order for the approximation for parsimonious modeling as in the work of Akaike through the well known Akaike Information Criterion (AIC, [1]) and the Final Prediction Error criterion [2,3]. Related issues are the estimation of the spectral density through AR models in the work of Parzen [29–31] or that of Priestley [6] which compares the performance of AR models to that of window based spectral estimators.

The results of Baxter [5] provide insights on the behavior of AR estimates as the model order goes to infinity. A useful result on the pointwise convergence of the AR parameters is presented in [17]. In the works of Pourahmadi [35,15,36] the nature and rates of convergence of the AR parameters have been discussed for univariate and multivariate stochastic processes. Related are the results discussed by Kreiss et al. [28] in the context of identifying applicability of the autoregressive sieve bootstrap. Similar results on the rate of convergence have been derived in [19,34].

In practice, the AR parameters are often derived using estimates of the covariance sequences and not the original covariance sequence of the process itself, as the latter is not readily available.[1] While studying the asymptotic behavior of such estimates, one has to impose conditions on the relationship between the number of samples N used for estimation and the order of estimation p. It was shown by Berk [7] that the AR parameters derived from an estimated covariance sequence converge in probability as long as $p = o\{N^{\frac{1}{3}}\}$.[2] Mean-square convergence of the AR parameters as the number of samples approaches infinity while the model order remains finite was studied in [8]. In [24], the authors have proved theorems relating to the rate of almost sure convergence of the covariance estimates to their true values and have subsequently derived the same for the AR parameters based on AR models driven by martingale difference innovation sequences. Their theorems require $p = O\{(\ln N)^{\alpha}\}$ for some $\alpha < \infty$. Among more recent works, [20,40,11] have studied the convergence of the estimated covariance matrix and AR parameters to their corresponding theoretical values under similar assumptions. However in most signal processing applications as well as in the simulation of Markov processes the assumption that the driving sequence is a martingale difference is too strong since all that can be guaranteed is stationarity of the underlying stochastic process and the use of resulting L_2 theory.

While there are a number of works concerned with the optimal model order; few results are available on the convergence of the spectral density of the approximating finite order AR process. The main motivation of this paper is to study the asymptotic behavior of the spectral density of finite order approximation models, as the order approaches infinity, and to obtain conditions under which the spectral density of the approximation converges to the spectral density of the infinite order AR estimate of the original process. We thereby identify a family of stochastic processes for which the spectral density of the original process may be derived from that of an approximating AR sequence.

We study the convergence of the spectral density at the origin as well as in L_2. These are motivated by issues arising in simulation and signal processing. The spectral density at the origin plays an important role in steady-state simulation due to the following invariance principle. Let $\{X_k\}$ be a WSS ergodic process with a finite spectral density at the origin and denote this quantity by Γ^2. This quantity is called the Time Average Variance Constant (TAVC) of the process [39]. If $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ is the sample mean, then according to the central limit theorem due to Ibragimov and Linnik [25],

$$\sqrt{n}(\bar{X}_n - \mu) \Longrightarrow \mathcal{N}(0, \Gamma^2)$$

where $\Longrightarrow$ denotes convergence in distribution.

Γ^2 is thus a quantity of importance in steady state simulations, where the objective is to compute the limit $\lim_{n\to\infty} \bar{X}_n$ when it exists [4]. One way of estimating Γ^2 is by windowed estimates based on finite order AR approximations of the observed stationary process. Instead of directly estimating the moments, one obtains approximations for the spectral density at the origin.

The organization of this paper is as follows. We begin with a study of the asymptotic behavior of the spectral density of MA estimates of a stationary time series. We show that when the sequence of the parameters of the Wold decomposition of the original process is summable, the spectral density converges to that of the original process in L_2.

Next, we study the asymptotic properties of the spectral density of AR estimates, considering first the case where AR parameters are derived based on knowing the true covariances of the original process. It is shown that when the spectral density of the process is strictly non-vanishing in $\left(-\frac{1}{2}, \frac{1}{2}\right]$ and its covariance sequence is summable, the spectral density of the approximating AR sequence converges in L_2, and also at the origin, as the order of approximation goes to infinity. In the

[1] We use the term "theoretical AR estimates" to refer to AR approximations based on solving the Yule–Walker equations using the true covariances and the term "empirical AR estimates" to refer to AR approximations based on empirical estimates of the covariance sequence.

[2] Standard notation:

$$k = o\{Z\} \Rightarrow \lim_{Z\to\infty} \frac{|k(Z)|}{|Z|} = 0$$

$$k = O\{Z\} \Rightarrow \lim_{Z\to\infty} \sup \frac{|k(Z)|}{|Z|} < \infty$$

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