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Extended matrix variate gamma and beta functions

Daya K. Nagar[a,*], Alejandro Roldán-Correa[a], Arjun K. Gupta[b]

[a] Instituto de Matemáticas, Universidad de Antioquia, Calle 67, No. 53–108, Medellín, Colombia

[b] Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, USA

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ABSTRACT

The gamma and beta functions have been generalized in several ways. The multivariate beta and multivariate gamma functions due to Ingham and Siegel have been defined as integrals having the integrand as a scalar function of the real symmetric matrix. In this article, we define extended matrix variate gamma and extended matrix variate beta functions thereby generalizing multivariate gamma and multivariate beta functions defined by Ingham and Siegel. We study a number of properties of these newly defined functions. We also give some applications of these functions to statistical distribution theory.

1. Introduction

The gamma function was first introduced by Leonard Euler in 1729, as the limit of a discrete expression and later as an absolutely convergent improper integral,

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\, \mathrm{d}t, \quad \mathrm{Re}(a) > 0. \tag{1}$$

The gamma function has many beautiful properties and has been used in almost all the branches of science and engineering.

One year later, Euler introduced the beta function defined for a pair of complex numbers a and b with positive real part through the integral

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, \mathrm{d}t, \quad \mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0. \tag{2}$$

The beta function has many properties, including the symmetry, $B(a, b) = B(b, a)$, and its relationship to the gamma function,

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

* Corresponding author.
E-mail addresses: dayaknagar@yahoo.com, dayaknagar@gmail.com (D.K. Nagar).

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In statistical distribution theory, gamma and beta functions have been used extensively. Using integrands of gamma and beta functions, the gamma and beta density functions are usually defined.

Recently, the domains of gamma and beta functions have been extended to the whole complex plane by introducing in the integrands of (1) and (2), the factors $\exp(-\sigma/t)$ and $\exp[-\sigma/t(1-t)]$, respectively, where $\mathrm{Re}(\sigma) > 0$. The functions so defined have been named the extended gamma and the extended beta functions.

In 1994, Chaudhry and Zubair [4] defined the extended gamma function, $\Gamma(a;\sigma)$, as

$$\Gamma(a;\sigma) = \int_0^\infty t^{a-1} \exp\left(-t - \frac{\sigma}{t}\right) \mathrm{d}t, \tag{3}$$

where $\mathrm{Re}(\sigma) > 0$ and a is any complex number. For $\mathrm{Re}(a) > 0$ and $\sigma = 0$, it is clear that the extension of the gamma function reduces to the classical gamma function, $\Gamma(a;0) = \Gamma(a)$. The extended gamma function is a special case of the Krätzel function defined in 1975 by Krätzel [19]. Further, this extension of the gamma function yields an interesting relationship with the modified Bessel function type 2 (Gradshteyn and Ryzhik [12]) as

$$\Gamma(\alpha;\sigma) = 2\sigma^{\alpha/2} K_\alpha(2\sqrt{\sigma}), \tag{4}$$

which may not be as apparent with the classical gamma function. The generalized gamma function (extended) has been proved very useful in various problems in engineering and physics, see for example, Chaudhry and Zubair [7,9,4,3,5,8,6].

In 1997, Chaudhry et al. [2] defined the extended beta function

$$B(a,b;\sigma) = \int_0^1 t^{a-1}(1-t)^{b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{5}$$

where $\mathrm{Re}(\sigma) > 0$ and parameters a and b are arbitrary complex numbers. When $\sigma = 0$, it is clear that for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$, the extended beta function reduces to the classical beta function $B(a,b)$. The extension of the beta function is related to the generalization of the gamma function by

$$\Gamma(x;\sigma)\Gamma(y;\sigma) = 2\int_0^\infty r^{2(x+y)-1} e^{-r^2} B\left(x, y; \frac{\sigma}{r^2}\right) \mathrm{d}r.$$

Furthermore, the extended gamma function has connections with error and Whittaker functions, and even give new representations for special cases of these functions. Using the extension of the beta function, Chaudhry et al. [2] also introduced an extended beta distribution.

Recently, Morán-Vásquez and Nagar [21] have applied the extended beta function in deriving certain probability distributions. The gamma function, the beta function, the gamma distribution and the beta distribution have been generalized to the matrix case in various ways. These generalizations and some of their properties can be found in Olkin [24], Gupta and Nagar [13], Mathai [20], Muirhead [22], and Nagar, Gupta, and Sánchez [23]. For some recent advances the reader is referred to Hassairi and Regaig [14], Farah and Hassairi [10], and Zine [26]. However, generalizations of extended gamma and beta functions to the matrix case have not been defined and studied. It is, therefore, of great interest to define generalizations of the extended gamma and the beta functions to the matrix case, study their properties, different integral representations, connections of these generalizations with other known special functions of the matrix argument.

This paper is divided into six sections. Section 2 deals with some well known definitions and results on matrix algebra, zonal polynomials and special functions of the matrix argument. In Section 3, the extended matrix variate gamma function has been defined and its properties have been studied. Definition and different integral representations of the extended matrix variate beta function are given in Section 4. Some integrals involving zonal polynomials and the extended matrix variate beta function are evaluated in Section 5. In Section 6, the distribution of the sum of two independent inverse Wishart matrices has been derived in terms of the extended matrix variate beta function. We introduce the extended matrix variate beta distribution in Section 7.

2. Some known definitions and results

In this section we give several known definitions and results. We first state the following notations and results that will be utilized in this and subsequent sections. Let $A = (a_{ij})$ be an $m \times m$ matrix of real or complex numbers. Then, A' denotes the transpose of A; $\mathrm{tr}(A) = a_{11} + \cdots + a_{mm}$; $\mathrm{etr}(A) = \exp(\mathrm{tr}(A))$; $\det(A) =$ determinant of A; $A = A' > 0$ means that A is symmetric positive definite, $0 < A < I_m$ means that both A and $I_m - A$ are symmetric positive definite, and $A^{1/2}$ denotes the unique positive definite square root of $A > 0$.

Several generalizations of Euler's gamma function are available in the scientific literature. The multivariate gamma function, which is frequently used in multivariate statistical analysis, is defined by (Ingham [16]),

$$\Gamma_m(a) = \int_{X>0} \mathrm{etr}(-X) \det(X)^{a-(m+1)/2} \,\mathrm{d}X, \tag{6}$$