Journal of Multivariate Analysis 122 (2013) 70–81

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

On the exact and approximate distributions of the product of a Wishart matrix with a normal vector

Taras Bodnar [a,*], Stepan Mazur [b], Yarema Okhrin [c]

[a] Department of Mathematics, Humboldt University of Berlin, D-10099 Berlin, Germany
[b] Department of Statistics, European University Viadrina, D-15207 Frankfurt (Oder), Germany
[c] Department of Statistics, University of Augsburg, D-86159 Augsburg, Germany

ARTICLE INFO

Article history:
Received 22 October 2012
Available online 27 July 2013

AMS 2000 subject classifications:
62H10
60E05

Keywords:
Wishart distribution
Multivariate normal distribution
Stochastic representation
Integral approximation

ABSTRACT

In this paper we consider the distribution of the product of a Wishart random matrix and a Gaussian random vector. We derive a stochastic representation for the elements of the product. Using this result, the exact joint density for an arbitrary linear combination of the elements of the product is obtained. Furthermore, the derived stochastic representation allows us to simulate samples of arbitrary size by generating independently distributed chi-squared random variables and standard multivariate normal random vectors for each element of the sample. Additionally to the Monte Carlo approach, we suggest another approximation of the density function, which is based on the Gaussian integral and the third order Taylor expansion. We investigate, with a numerical study, the properties of the suggested approximations. A good performance is documented for both methods.

1. Introduction

The basic building block of classical multivariate analysis is the multivariate normal distribution. Its properties are very well understood. Unlike the normal distribution, the theory of the Wishart distribution is less established, but nevertheless, contains numerous important and useful results. Many important distributional properties of Wishart matrices, inverse Wishart matrices, and related statistics are discussed in detail by [9,1,6], and others. The characterization of the Wishart distribution is presented in [16], who extended the results of [7,8], while [13–15] considered the generalization of the Wishart distribution constructed as a quadratic form of a T-distributed random matrix (cf. [5]) whose density function is based on the beta function of the matrix argument (see, e.g., [12]).

The joint distribution functions of the multivariate normal and the Wishart distributions have not been extensively studied in the statistical literature. There are mainly results on the distribution of quadratic forms (see e.g. [17]), despite the fact that the product of the (inverse) Wishart distribution and the normal distribution appears in many applications. A classical example can be found in discriminant analysis, where the elements of the discriminant function are computed as products of the inverse sample covariance matrix multiplied by the sample mean vector. Another important example is taken from portfolio theory in finance, where the weights of the tangency portfolio are estimated by the same product using historical asset returns (see e.g., [4,2]). If frequentist methods are used in analyzing the distributional properties of the discriminant function and/or of the estimated portfolio weights, then we have to deal with the product of the inverse Wishart matrix and a normal vector. This issue was investigated recently by [2]. However, in the Bayesian framework, we obtain the inverse Wishart distribution as the posterior distribution of the sample covariance matrix. This leads to the product of the Wishart matrix and a normal vector. Obtaining the distributional properties of this product statistic is the main goal of the present paper.

* Corresponding author.
E-mail address: bodnar@math.hu-berlin.de (T. Bodnar).

0047-259X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmva.2013.07.007

In this paper we investigate the Wishart distribution in combination with a Gaussian vector. In particular, we consider the expressions which depend on $\mathbf{Az}$, where $\mathbf{A}$ is a Wishart matrix and $\mathbf{z}$ is a Gaussian vector, which are independently distributed. First, we derive a stochastic representation and the exact density function of $\mathbf{LAz}$ for an arbitrary deterministic matrix $\mathbf{L}$. Second, we consider two important special cases. In the first example it is assumed that the covariance matrix $\mathbf{\Sigma}$ is equal to the identity matrix and $\mathbf{L}$ is a vector in the second case. In both cases the stochastic representations and the exact densities are derived. Moreover, we suggest a further approximation of the density of $\mathbf{LAz}$ which is based on the Gaussian integral and the third order Taylor series expansion. The performance of the approximate densities is analyzed with an extensive Monte Carlo study.

The rest of the paper is structured as follows. The main results are presented in Section 2, where the stochastic representation for the product $\mathbf{LAz}$ is derived as Theorem 1. It is applied to derive the density function in Corollary 1. Several important special cases are considered in Corollaries 2 and 3. In Section 2.1 we find an approximation for the density of $\mathbf{LAz}$ that is based on the third order Taylor series approximation (Theorem 2). The results of numerical studies are given in Section 3, while Section 4 summarizes the paper. The Appendix contains the Proof of Theorem 2.

2. Main results

Let $\mathbf{A}$ be a k-dimensional Wishart matrix with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$, that is, $\mathbf{A} \sim W_k(n, \mathbf{\Sigma})$. We assume that $n > k$, implying that the matrix $\mathbf{A}$ is non-singular. Furthermore, let $\mathbf{z} \sim N_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$, i.e., it follows a k-dimensional multivariate normal distribution. Throughout the paper it is assumed that $\lambda > 0$ and that $\mathbf{\Sigma}$ is positive definite. Let $\stackrel{d}{=}$ denote equality in distribution and $\mathbf{I}_p$ stand for the identity matrix of order p. In Theorem 1 we present a stochastic representation for p linear combinations of the elements of the random vector $\mathbf{Az}$, that is, $\mathbf{LAz}$, where $\mathbf{L}$ is a $p \times k$ constant matrix of rank $p < k$. The distribution of the product is given in terms of a χ^2 random variable and of two standard multivariate normal random vectors which are independently distributed. Stochastic representation is a very powerful tool in multivariate statistics. It plays an important role in the theory of elliptically contoured distributions (cf. [10]) and is widely used in Monte Carlo simulations. In particular, the simulation of the values of the product is considerably simplified if we use the stochastic representation and not the original definition based on the multivariate normal samples.

Theorem 1. *Let* $\mathbf{A} \sim W_k(n, \mathbf{\Sigma})$, $\mathbf{z} \sim N_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$ *with* $\lambda > 0$ *and* $\mathbf{\Sigma}$ *is positive definite. Assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independent. Let* $\mathbf{L}$ *be a* $p \times k$ *constant matrix of rank* $p < k$ *and let* $\mathbf{S}_1 = (\mathbf{L\Sigma L}^T)^{-1/2}\mathbf{L\Sigma}^{1/2}$, $\mathbf{S}_2 = (\mathbf{I}_p - \mathbf{S}_1^T\mathbf{S}_1)^{1/2}$. *Then the stochastic representation of* $\mathbf{LAz}$ *is given by*

$$\mathbf{LAz} \stackrel{d}{=} \xi(\mathbf{L\Sigma L}^T)^{1/2}\mathbf{y}_1 + \sqrt{\xi}(\mathbf{L\Sigma L}^T)^{1/2}\left[\sqrt{\mathbf{y}_1^T\mathbf{y}_1+\eta}\,\mathbf{I}_p - \frac{\sqrt{\mathbf{y}_1^T\mathbf{y}_1+\eta}-\sqrt{\eta}}{\mathbf{y}_1^T\mathbf{y}_1}\mathbf{y}_1\mathbf{y}_1^T\right]\mathbf{z}_0, \tag{1}$$

where $\xi \sim \chi^2_n$, $\mathbf{z}_0 \sim N_p(\mathbf{0}, \mathbf{I}_p)$,

$$\mathbf{y} = \begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} \sim N_k\left(\begin{pmatrix}\mathbf{S}_1\mathbf{\Sigma}^{1/2}\boldsymbol{\mu}\\ \mathbf{S}_2\mathbf{\Sigma}^{1/2}\boldsymbol{\mu}\end{pmatrix}, \lambda\begin{pmatrix}\mathbf{S}_1\mathbf{\Sigma}^2\mathbf{S}_1^T & \mathbf{S}_1\mathbf{\Sigma}^2\mathbf{S}_2^T\\ \mathbf{S}_2\mathbf{\Sigma}^2\mathbf{S}_1^T & \mathbf{S}_2\mathbf{\Sigma}^2\mathbf{S}_2^T\end{pmatrix}\right) \quad \text{with } \eta = \mathbf{y}_2^T\mathbf{y}_2;$$

ξ *and* $\mathbf{z}_0$ *are independent of* $\mathbf{y}$.

Proof. Since $\mathbf{A}$ and $\mathbf{z}$ are independently distributed, it follows that the conditional distribution of $\mathbf{LAz}|(\mathbf{z} = \mathbf{z}^*)$ is equal to the distribution of $\mathbf{LAz}^*$. Let $\tilde{\mathbf{L}} = (\mathbf{L}^T, \mathbf{z}^*)^T$. Then $\tilde{\mathbf{A}} = \tilde{\mathbf{L}}\mathbf{A}\tilde{\mathbf{L}}^T = \{\tilde{\mathbf{A}}_{ij}\}_{i,j=1,2}$ can be partitioned with $\tilde{\mathbf{A}}_{11} = \mathbf{LAL}^T$, $\tilde{\mathbf{A}}_{12} = \mathbf{LAz}^*$, $\tilde{\mathbf{A}}_{21} = \mathbf{z}^{*T}\mathbf{AL}^T$ and $\tilde{A}_{22} = \mathbf{z}^{*T}\mathbf{Az}^*$. Similarly, $\mathbf{H} = \tilde{\mathbf{L}}\mathbf{\Sigma}\tilde{\mathbf{L}}^T = \{\mathbf{H}_{ij}\}_{i,j=1,2}$ with $\mathbf{H}_{11} = \mathbf{L\Sigma L}^T$, $\mathbf{H}_{12} = \mathbf{L\Sigma z}^*$, $\mathbf{H}_{21} = \mathbf{z}^{*T}\mathbf{\Sigma L}^T$ and $H_{22} = \mathbf{z}^{*T}\mathbf{\Sigma z}^*$.

Because $\mathbf{A} \sim W_k(n, \mathbf{\Sigma})$ and $\text{rank}(\tilde{\mathbf{L}}) = p + 1 \le k$, we get from Theorem 3.2.5 of [18] that $\tilde{\mathbf{A}} \sim W_{p+1}(n, \mathbf{H})$. Furthermore, the application of Theorem 3.2.10 of [18] leads to

$$\tilde{\mathbf{A}}_{12}|\tilde{A}_{22}, \mathbf{z} = \mathbf{z}^* \sim N_p(\mathbf{H}_{12}H_{22}^{-1}\tilde{A}_{22}, \mathbf{H}_{11\cdot 2}\tilde{A}_{22}), \tag{2}$$

where $\mathbf{H}_{11\cdot 2} = \mathbf{H}_{11} - \mathbf{H}_{12}H_{22}^{-1}\mathbf{H}_{21}$.

Let $\xi = \tilde{A}_{22}/H_{22}$. Then

$$\mathbf{LAz}\left|\left[\frac{\mathbf{z}^T\mathbf{Az}}{\mathbf{z\Sigma z}} = \xi, \mathbf{z}\right]\right. \sim N_p(\xi\mathbf{L\Sigma z}, \xi(\mathbf{z}^T\mathbf{\Sigma z L\Sigma L}^T - \mathbf{L\Sigma zz}^T\mathbf{\Sigma L}^T)). \tag{3}$$

Because ξ and $\mathbf{z}$ are independent (cf., [18, Theorem 3.2.8]) the stochastic representation of $\mathbf{LAz}$ is

$$\mathbf{LAz} \stackrel{d}{=} \xi\mathbf{L\Sigma z} + \sqrt{\xi}(\mathbf{z}^T\mathbf{\Sigma zL\Sigma L}^T - \mathbf{L\Sigma zz}^T\mathbf{\Sigma L}^T)^{1/2}\mathbf{z}_0,$$

where $\xi \sim \chi^2_n$, $\mathbf{z}_0 \sim N_p(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{z} \sim N_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$; ξ, $\mathbf{z}_0$, and $\mathbf{z}$ are mutually independent.