Journal of Multivariate Analysis 122 (2013) 409–413

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Note(s)

Extremal t processes: Elliptical domain of attraction and a spectral representation

T. Opitz

I3M (CC51), Université Montpellier 2, Place Eugène Bataillon, 34095 Montpellier cedex 5, France

ARTICLE INFO

Article history:
Received 29 October 2012
Available online 20 August 2013

AMS 2000 subject classifications:
60G70
60G60

Keywords:
Elliptical distribution
Extremal t process
Max-stable process
Spectral construction

ABSTRACT

The extremal t process was proposed in the literature for modeling spatial extremes within a copula framework based on the extreme value limit of elliptical t distributions (Davison et al. (2012) [5]). A major drawback of this max-stable model was the lack of a spectral representation such that for instance direct simulation was infeasible. The main contribution of this note is to propose such a spectral construction for the extremal t process. Interestingly, the extremal Gaussian process introduced by Schlather (2002) [22] appears as a special case. We further highlight the role of the extremal t process as the maximum attractor for processes with finite-dimensional elliptical distributions. All results naturally also hold within the multivariate domain.

1. Introduction

Davison et al. [5] survey the statistical modeling of spatial extremes and provide a global view on available models and their interconnections. Among these models, the extremal t process represents a max-stable process that generalizes the t extreme value copula to infinite dimension. It is well defined, yet no direct construction was known back then which led the authors to class it among copula models characterized by their motivation from multivariate considerations. An application to Swiss rainfall data in that paper bears witness of its versatility for extremal dependence modeling. In the following, we show that the extremal t process provides a natural connection between two prominent max-stable model classes, namely Schlather's extremal Gaussian process [22] and the Brown–Resnick process defined in [3] and revisited in a more general context in [17]. The connection to the Brown–Resnick process was detailed for the multivariate context in [20] and is related to the study of elliptical triangular arrays with the Hüsler–Reiss distribution [15,11] as the maximum attractor [12]; see also [14] who show that the triangular arrays of χ^2-distributed random vectors lead to the same maximum attractor. This connection was then interpreted for the spatial context in [5]. The extremal t dependence structure is further proposed for semi-parametric inference in a multivariate context by Klüppelberg et al. [18,19]. We conceive a spectral representation for the extremal t process that generalizes the one of the extremal Gaussian process. It renders direct simulation possible for moderately large general degrees of freedom.

The remainder of the paper is organized as follows: Section 2 gives some background in extreme value theory and reviews the results for elliptical distributions. Spectral constructions of multivariate extremal t distributions and extremal t processes are presented in Section 3, along with a statement on the domain of attraction for processes with finite-dimensional elliptical distributions. We conclude with a discussion and potential future developments in Section 4.

The following notational conventions shall apply in the remainder of the paper: vectors are typeset in bold face, in particular the vector constants $\mathbf{0} = (0, \ldots, 0)^T$ and $\mathbf{1} = (1, \ldots, 1)^T$. If not stated otherwise, operations on vectorial arguments

E-mail address: thomas.opitz@math.univ-montp2.fr.
URL: http://www.ens.math.univ-montp2.fr/~opitz/.

0047-259X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmva.2013.08.008

like maxima or arithmetic operations must be interpreted componentwise, as for instance in $\mathbf{a}^{-1} = (a_1^{-1}, \dots, a_d^{-1})^T$. Inequalities apply componentwise: $\mathbf{a} > \mathbf{b}$ means that $a_j > b_j$ for $j = 1, \dots, d$ whereas $\mathbf{a} \not\leq \mathbf{b}$ means that $a_j > b_j$ for at least one j. Rectangular bounded or unbounded sets are given according to notations like $[\mathbf{u}, \mathbf{v}] = [u_1, v_1] \times \cdots \times [u_d, v_d]$ or $(\mathbf{0}, \infty) = (0, \infty) \times \cdots \times (0, \infty)$. The complementary set of a set B in $\mathbb{R}^d$ is written B^c. The truncation operator $x^+ = \max(x, 0)$ maps the negative values to 0. The indicator function of a set B is denoted by $\chi_B(\cdot)$.

2. Extreme value theory

For a more detailed account of max-stability and extreme value theory in general we refer the reader to the textbooks of [2,7].

2.1. Max-stability

Let $\mathbf{Z}, \mathbf{Z_1}, \mathbf{Z_2}, \dots$ be a sequence of independent and identically distributed (iid) random vectors in $\mathbb{R}^d$ ($d \geq 1$) with nondegenerate univariate marginal distributions. We say that $\mathbf{Z}$ follows a max-stable distribution G if sequences of normalizing vectors $\mathbf{a_n} > \mathbf{0}$ and $\mathbf{b_n}$ ($n = 1, 2, \dots$) exist such that the equality in distribution

$$\max_{i=1,\dots,n} \mathbf{a_n}^{-1}(\mathbf{Z_i} - \mathbf{b_n}) \stackrel{d}{=} \mathbf{Z} \sim G \tag{1}$$

holds for the componentwise maximum. A full characterization of multivariate max-stable distributions leads to rather technical expressions. For our purposes, it is convenient to focus on common α-Fréchet marginal distributions $G_j(z_j) = \Phi_\alpha(z_j) = \exp(-z_j^{-\alpha})\chi_{(0,\infty)}(z_j)$ ($j = 1, \dots, d$) for some *tail index* $\alpha > 0$. Monotone and parametric marginal transformations allow reconstructing all admissible univariate max-stable marginal scales in (1) from this particular marginal scale. More precisely, the class of univariate max-stable distributions is partitioned into the class of α-Fréchet distributions under strictly increasing linear transformations and further the so-called Gumbel and Weibull classes.

With α-Fréchet marginal distributions, the *standard exponent measure* $\mathbb{M}$ can be defined on $[\mathbf{0}, \infty) \setminus \{\mathbf{0}\}$ by $\mathbb{M}(([\mathbf{0}, \mathbf{z}]^c) = -\log \mathbb{P}(\mathbf{Z}^\alpha \leq \mathbf{z})$ with the convention $-\log 0 = \infty$ and characterizes the dependence structure in G on a standardized scale; it is uniquely defined by the dependence function $M(\mathbf{z}) = \mathbb{M}([\mathbf{0}, \mathbf{z}]^c)$ which takes the value ∞ whenever $\min_j z_j = 0$ such that $\mathbf{z} \notin (\mathbf{0}, \infty)$. The extremal coefficient $M(\mathbf{1}) \in [1, d]$ can serve as an indicator of the strength of extremal dependence, ranging from full dependence associated with the value 1 to independence associated with the value d (cf. [23]).

In the infinite-dimensional domain, we call a stochastic process $\mathbf{Z} = \{Z(s), s \in S \subset \mathbb{R}^p\}$ ($p \geq 1$) with a non-empty Borel set S max-stable if its finite-dimensional distributions are max-stable. If $\mathbf{Z_1}, \mathbf{Z_2}, \dots$ are iid copies of $\mathbf{Z}$, then sequences of functions $a_n(s) > 0$ and $b_n(s)$ ($n \geq 1$) exist such that $\{\max_{i=1,\dots,n} a_n(s)^{-1}(Z_i(s) - b_n(s))\} \stackrel{d}{=} \{Z(s)\}$.

2.2. Domain of attraction

Let $\mathbf{X}, \mathbf{X_1}, \mathbf{X_2}, \dots$ be a sequence of iid random vectors in $\mathbb{R}^d$ with distribution function F. For suitably chosen normalizing sequences, relation (1) can hold asymptotically in the sense of distributional convergence with nondegenerate marginal distributions in the limit $\mathbf{Z}$:

$$\max_{i=1,\dots,n} \mathbf{a_n}^{-1}(\mathbf{X_i} - \mathbf{b_n}) \stackrel{d}{\to} \mathbf{Z} \quad (n \to \infty). \tag{2}$$

We say that the distribution F of $\mathbf{X}$ is in the max-domain of attraction (MDA) of the max-stable distribution G of $\mathbf{Z}$, or simply that $\mathbf{X}$ is in the MDA of $\mathbf{Z}$. Normalizing sequences are not unique and the limit distribution G is unique up to a linear transformation. If normalizing constants can be chosen such that all the univariate marginal distributions G_j are of the same α-Fréchet type, then the particular choice of $\mathbf{b_n} = \mathbf{0}$ is admissible. In this case, the convergence in distribution (2) is equivalent to

$$n\,\mathbb{P}(\mathbf{a_n}^{-1}\mathbf{X} \not\leq \mathbf{z}) \to M(\mathbf{z}^\alpha) \quad \text{for all } \mathbf{z} \in (\mathbf{0}, \infty). \tag{3}$$

For $d = 1$, we have $n\,\mathbb{P}(a_n^{-1}X \geq z) \to z^{-\alpha}$ ($z > 0$), and then X is said to be *regularly varying at* ∞ *with index* $\alpha > 0$ or just *regularly varying* in the remainder of this paper, denoted as $X \in \mathrm{RV}_\alpha$. The normalizing sequence can be chosen as $a_n = \inf\{x : \mathbb{P}(X \geq x) \leq n^{-1}\}$.

For stochastic processes, the notion of MDA is defined in the sense of the convergence of all finite-dimensional distributions according to (2).

2.3. A spectral representation for max-stable processes

The commonly used models for max-stable processes are generated with so-called spectral constructions whose first appearance dates back to the seminal paper of [6]. Schlather [22] proposes to use a Poisson process $\{V_i\} \sim \mathrm{PRM}(v^{-2}dv)$ on