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Archimedean survival processes

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ABSTRACT

Archimedean copulas are popular in the world of multivariate modelling as a result of their breadth, tractability, and flexibility. McNeil and Nešlehová (2009) [12] showed that the class of Archimedean copulas coincides with the class of positive multivariate ℓ_1-norm symmetric distributions. Building upon their results, we introduce a class of multivariate Markov processes that we call 'Archimedean survival processes' (ASPs). An ASP is defined over a finite time interval, is equivalent in law to a vector of independent gamma processes, and its terminal value has an Archimedean survival copula. There exists a bijection from the class of ASPs to the class of Archimedean copulas. We provide various characterisations of ASPs, and a generalisation.

1. Introduction

The use of copulas has become commonplace for dependence modelling in finance, insurance, and risk management (see, for example, [4,6,11]). From a modelling perspective, one of the attractive features of copulas is that they allow the fitting of one-dimensional marginal distributions to be performed separately from the fitting of cross-sectional dependence.

The Archimedean copulas—a subclass of copulas—have received particular attention in the literature for both their tractability and practical convenience (see, for example, [7,8] and [14, Chapter 4]). We introduce a family of multivariate stochastic processes that we call *Archimedean survival processes* (ASPs). ASPs are constructed in such a way that they are naturally linked to Archimedean copulas. An ASP is defined over a finite time horizon, and its terminal value has a multivariate ℓ_1-norm symmetric distribution. This implies that the terminal value of an ASP has an Archimedean survival copula. Indeed, there is a bijection from the class of Archimedean copulas to the class of ASPs.

Norberg [15] suggested using a randomly-scaled gamma bridge (also called a Dirichlet process) for modelling the cumulative payments made on insurance claims (see also [3]). Such a process $\{\xi_{tT}\}_{0\leq t\leq T}$ can be constructed as $\xi_{tT} = R\gamma_{tT}$, where R is a positive random variable independent of a gamma bridge $\{\gamma_{tT}\}$ satisfying $\gamma_{0T} = 0$ and $\gamma_{TT} = 1$, for some $T \in (0, \infty)$. This is an increasing process and so lends itself to the modelling of cumulative gains or losses; in this case the random variable R represents the total, final gain. We can interpret R as a signal and the gamma bridge $\{\gamma_{tT}\}$ as independent multiplicative noise. The process $\{\xi_{tT}\}$ can be considered to be a gamma process conditioned so that ξ_{TT} has the law of R, and so belongs to the class of Lévy random bridges (see [10]). As such, we call the process $\{\xi_{tT}\}$ a 'gamma random bridge' (GRB).

ASPs are an n-dimensional extension of gamma random bridges. Each one-dimensional marginal process $\{\xi_t^{(i)}\}$ of an ASP $\{(\xi_t^{(1)}, \ldots, \xi_t^{(n)})^\top\}_{0\leq t\leq T}$ is a GRB. We shall construct each $\{\xi_t^{(i)}\}$ by splitting a 'master' GRB into n non-overlapping

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subprocesses. This method of splitting a Lévy random bridge into subprocesses (which are themselves Lévy random bridges) was used by Hoyle et al. [9] to develop a bivariate insurance reserving model based on random bridges of the stable-1/2 subordinator. A remarkable feature of the proposed construction is that the terminal vector $(\xi_T^{(1)}, \dots, \xi_T^{(n)})^\top$ has a multivariate ℓ_1-norm symmetric distribution, and hence an Archimedean survival copula.

We shall also construct *Liouville processes* by splitting a GRB into n pieces. By allowing more flexibility in the splitting mechanism and by employing some deterministic time changes, a broader range of behaviour can be achieved by Liouville processes than ASPs. For example, the one-dimensional marginal processes of a Liouville process are in general not identical in law.

A direct application of ASPs and Liouville processes is to the modelling of multivariate cumulative gain (or loss) processes. Consider, for example, an insurance company that underwrites several lines of motor business (such as personal motor, fleet motor or private-hire vehicles) for a given accident year. A substantial payment made on one line of business is unlikely to coincide with a substantial payment made on another line of business (e.g. a large payment is unlikely to be made on a personal motor claim at the same time as a large payment is made on a fleet motor claim). However, the total sums of claims arising from the lines of business will depend on certain common factors such as prolonged periods of adverse weather or the quality of the underwriting process at the company. Such common factors will produce dependence across the lines. An ASP or a Liouville process might be a suitable model for the cumulative paid-claims processes of the lines of motor business. The one-dimensional marginal processes of a Liouville process are increasing and do not exhibit simultaneous large jumps, but they can display strong correlation.

ASPs can be used to interpolate a dependence structure when using Archimedean copulas in discrete-time models. Consider a risk model where the marginal distributions of the returns on n assets are fitted for the future dates $t_1 < \cdots < t_n < T < \infty$. An Archimedean copula C is used to model the dependence of the returns to time T. At this stage we have a model for the joint distribution of returns to time T, but we have only the one-dimensional marginal distributions at the intertemporal times $t_1, \dots, t_n$. The problem then is to choose copulas to complete the joint distributions of the returns to the times $t_1, \dots, t_n$ in a way that is consistent with the time-T joint distribution. For each time t_i, this can be achieved by using the time-t_i survival copula implied by the ASP with survival copula C at terminal time T.

This paper is organised as follows: In Section 2, we review multivariate ℓ_1-norm symmetric distributions, multivariate Liouville distributions, Archimedean copulas and gamma random bridges. In Section 3, we define ASPs and provide various characterisations of their law. We detail how to construct a multivariate process such that each one-dimensional marginal is uniformly distributed. An application is then given where an ASP is used to solve an Archimedean copula interpolation problem. In Section 4, we generalise ASPs to Liouville processes. We apply Liouville processes to the intraday forecasting of realised variance. In Section 5, we state our conclusions.

2. Preliminaries

We fix a probability space $(\Omega, \mathrm{P}, \mathcal{F})$ and assume that all processes under consideration are càdlàg, and all filtrations are right-continuous. We let f^{-1} denote the generalised inverse of a monotonic function f. Thus, if f is decreasing then $f^{-1}(y) = \inf\{x : f(x) \le y\}$. We denote the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$ by $\|\mathbf{x}\|$, i.e. $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$.

We present some definitions and results from the theory of multivariate distributions and refer the reader to Fang et al. [5] for further details.

Let $\mathbf{G}$ be a vector of independent random variables such that G_i is a gamma random variable with shape parameter $\alpha_i > 0$ and unit scale parameter. Then the random vector $\mathbf{D} = \mathbf{G}/\|\mathbf{G}\|$ has a *Dirichlet distribution* with *parameter vector* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^\top$. In two dimensions, a Dirichlet random variable can be written as $(B, 1 - B)^\top$, where B is a beta random variable. If all the elements of the parameter vector $\boldsymbol{\alpha}$ are identical, then $\mathbf{D}$ is said to have a *symmetric* Dirichlet distribution.

A random variable $\mathbf{X}$ taking values in $\mathbb{R}^n$ has a *multivariate Liouville distribution* if $\mathbf{X} \overset{\text{law}}{=} R\mathbf{D}$, for $R \ge 0$ a random variable, and $\mathbf{D}$ a Dirichlet random variable, independent of R, with parameter vector $\boldsymbol{\alpha}$. We call the law of R the *generating law* and $\boldsymbol{\alpha}$ the *parameter vector* of the distribution. In the case where R is positive and has a density p, the density of $\mathbf{X}$ exists and can be written as

$$\mathbf{x} \mapsto \Gamma(\|\boldsymbol{\alpha}\|) \frac{p(\|\mathbf{x}\|)}{(\|\mathbf{x}\|)^{\|\boldsymbol{\alpha}\|-1}} \prod_{i=1}^n \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \tag{1}$$

for $\mathbf{x} \in \mathbb{R}_+^n$, where $\Gamma(x)$ is the gamma function [1, 6.1]. In the case $\boldsymbol{\alpha} = (1, \dots, 1)^\top$, $\mathbf{X}$ has a *multivariate ℓ_1-norm symmetric distribution*. A multivariate ℓ_1-norm symmetric distribution is characterised by its generating law.

McNeil and Nešlehová [12] give an account of how Archimedean copulas coincide with survival copulas of ℓ_1-norm symmetric distributions which have no point-mass at the origin. Then in [13], McNeil and Nešlehová generalise Archimedean copulas to so-called Liouville copulas, which are defined as the survival copulas of multivariate Liouville distributions.

A copula is a distribution function on the unit hypercube with the added property that each one-dimensional marginal distribution is uniform. Archimedean copulas are copulas that take a particular functional form. The following definition given in [12] is convenient for the present work: A decreasing and continuous function $\psi : [0, \infty) \to [0, 1]$ which satisfies