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# Further results on the *h-test* of Durbin for stable autoregressive processes

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ABSTRACT

The purpose of this paper is to investigate the asymptotic behavior of the Durbin–Watson statistic for the stable $p$-order autoregressive process when the driven noise is given by a first-order autoregressive process. It is an extension of the previous work of Bercu and Proïa devoted to the particular case $p = 1$. We establish the almost sure convergence and the asymptotic normality for both the least squares estimator of the unknown vector parameter of the autoregressive process as well as for the serial correlation estimator associated with the driven noise. In addition, the almost sure rates of convergence of our estimates are also provided. Then, we prove the almost sure convergence and the asymptotic normality for the Durbin–Watson statistic and we derive a two-sided statistical procedure for testing the presence of a significant first-order residual autocorrelation that appears to simplify and to improve the well-known *h-test* suggested by Durbin. Finally, we briefly summarize our observations on simulated samples.



## 1. Introduction

The Durbin–Watson statistic was originally introduced by the eponymous econometricians Durbin and Watson [16–18] in the middle of last century, in order to detect the presence of a significant first-order autocorrelation in the residuals from a regression analysis. The statistical test worked pretty well in the independent framework of linear regression models, as it was specifically investigated by Tillman [35]. While the Durbin–Watson statistic started to become well-known in Econometrics by being commonly used in the case of linear regression models containing lagged dependent random variables, Malinvaud [28] and Nerlove and Wallis [30] observed that its widespread use in inappropriate situations were leading to inadequate conclusions. More precisely, they noticed that the Durbin–Watson statistic was asymptotically biased in the dependent framework. To remedy this misuse, alternative compromises were suggested. In particular, Durbin [14] proposed a set of revisions of the original test, as the so-called *t-test* and *h-test*, and explained how to use them focusing on the first-order autoregressive process. It inspired a lot of works afterwards. More precisely, Maddala and Rao [27], Park [31] and then Inder [23,24] and Durbin [15] looked into the approximation of the critical values and distributions under the null hypothesis, and showed by simulations that alternative tests significantly outperformed the inappropriate one, even on small-sized samples. Additional improvements were brought by King and Wu [25] and lately, Stocker [32] gave substantial contributions to the study of the asymptotic bias resulting from the presence of lagged dependent random variables. In

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most cases, the first-order autoregressive process was used as a reference for related research. This is the reason why the recent work of Bercu and Proïa [4] was focused on such a process in order to give a new light on the distribution of the Durbin–Watson statistic under the null hypothesis as well as under the alternative hypothesis. They provided a sharp theoretical analysis rather than Monte Carlo approximations, and they proposed a statistical procedure derived from the Durbin–Watson statistic. They showed how, from a theoretical and a practical point of view, this procedure outperforms the commonly used Box–Pierce [7] and Ljung–Box [6] statistical tests, in the restrictive case of the first-order autoregressive process, even on small-sized samples. They also explained that such a procedure is asymptotically equivalent to the *h-test* of Durbin [14] for testing the significance of the first-order serial correlation. This work [4] had the ambition to bring the Durbin–Watson statistic back into light. It also inspired Bitseki Penda, Djellout and Proïa [5] who established moderate deviation principles on the least squares estimators and the Durbin–Watson statistic for the first-order autoregressive process where the driven noise is also given by a first-order autoregressive process.

Our goal is to extend of the previous results of Bercu and Proïa [4] to $p$-order autoregressive processes, contributing moreover to the investigation on several open questions left unanswered during four decades on the Durbin–Watson statistic [14,15,30]. One will observe that the multivariate framework is much more difficult to handle than the scalar case of [4]. We will focus our attention on the $p$-order autoregressive process given, for all $n \geq 1$, by

$$\begin{cases} X_n = \theta_1 X_{n-1} + \cdots + \theta_p X_{n-p} + \varepsilon_n \\ \varepsilon_n = \rho \varepsilon_{n-1} + V_n \end{cases} \tag{1.1}$$

where the unknown parameter $\theta = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_p \end{pmatrix}'$ is a nonzero vector such that $\|\theta\|_1 < 1$, and the unknown parameter $|\rho| < 1$. Via an extensive use of the theory of martingales [12,21], we shall provide a sharp and rigorous analysis on the asymptotic behavior of the least squares estimators of $\theta$ and $\rho$. The previous results of convergence were first established in probability [28,30], and more recently almost surely [4] in the particular case where $p = 1$. We shall prove the almost sure convergence as well as the asymptotic normality of the least squares estimators of $\theta$ and $\rho$ in the more general multivariate framework, together with the almost sure rates of convergence of our estimates. We will deduce the almost sure convergence and the asymptotic normality for the Durbin–Watson statistic. Therefore, we shall be in the position to propose further results on the well-known *h-test* of Durbin [14] for testing the significance of the first-order serial correlation in the residuals. We will also explain why, on the basis of the empirical power, this test procedure outperforms Ljung–Box [6] and Box–Pierce [7] *portmanteau* tests for stable autoregressive processes. We will finally show by simulation that it is equally powerful than the Breusch–Godfrey [8,19] test and the *h-test* [14] on large samples, and better than all of them on small samples.

The paper is organized as follows. Section 2 is devoted to the estimation of the autoregressive parameter. We establish the almost sure convergence of the least squares vector estimator of $\theta$ to the limiting value

$$\theta^* = \alpha \left( I_p - \theta_p \rho J_p \right) \beta \tag{1.2}$$

where $I_p$ is the identity matrix of order $p$, $J_p$ is the exchange matrix of order $p$, and where $\alpha$ and $\beta$ will be calculated explicitly. The asymptotic normality as well as the quadratic strong law and a set of results derived from the law of iterated logarithm are provided. Section 3 deals with the estimation of the serial correlation parameter. The almost sure convergence of the least squares estimator of $\rho$ to

$$\rho^* = \theta_p \rho \theta_p^* \tag{1.3}$$

where $\theta_p^*$ stands for the $p$-th component of $\theta^*$ is also established along with the quadratic strong law, the law of iterated logarithm and the asymptotic normality. It enables us to establish in Section 4 the almost sure convergence of the Durbin–Watson statistic to

$$D^* = 2(1 - \rho^*) \tag{1.4}$$

together with its asymptotic normality. Our sharp analysis on the asymptotic behavior of the Durbin–Watson statistic remains true whatever the values of the parameters $\theta$ and $\rho$ as soon as $\|\theta\|_1 < 1$ and $|\rho| < 1$, assumptions resulting from the stability of the model. Consequently, we are able in Section 4 to propose a two-sided statistical test for the presence of a significant first-order residual autocorrelation closely related to the *h-test* of Durbin [14]. A theoretical comparison as well as a sharp analysis of both approaches are also provided. In Section 5, we give a short conclusion where we briefly summarize our observations on simulated samples. We compare the empirical power of this test procedure with the commonly used *portmanteau* tests of Box–Pierce [7] and Ljung–Box [6], with the Breusch–Godfrey test [8,19] and the *h-test* of Durbin [14]. Finally, the proofs related to linear algebra calculations are postponed to Appendix A and all the technical proofs of Sections 2 and 3 are postponed to Appendices B and C, respectively. Moreover, Appendix D is devoted to the asymptotic equivalence between the *h-test* of Durbin and our statistical test procedure.

**Remark 1.1.** In the whole paper, for any matrix $M$, $M'$ is the transpose of $M$. For any square matrix $M$, $\mathrm{tr}(M)$, $\det(M)$, $|||M|||_1$ and $\rho(M)$ are the trace, the determinant, the 1-norm and the spectral radius of $M$, respectively. In addition, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the smallest and the largest eigenvalues of $M$, respectively. For any vector $v$, $\|v\|$ stands for the euclidean norm of $v$ and $\|v\|_1$ is the 1-norm of $v$.