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The distribution of the amplitude and phase of the mean of a sample of complex random variables

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ABSTRACT

Edgeworth-type expansions are given for the distribution of (normalized versions of) the amplitude and phase of the mean of a sample of complex random variables. These expansions are transformed to polar forms with applications to modeling signals from a cell-phone. Limiting distributions of (normalized versions of) the amplitude and phase of the mean are given for the cases: (1) population mean is zero, and (2) population mean is non-zero. The results apply for distributions with finite cumulants.

1. Introduction and summary

The need for the distribution of the amplitude and phase of the mean of a sample of complex random variables arises in many applied areas. We mention: statistical physics, signal processing, circuits and systems, oceanography, microwave theory and techniques, aerospace and electronic systems, military communications, waves in random media, acoustics, speech and audio processing, optics, information theory, photonics technology, nuclear energy, solid-state electronics, ultrasonics and frequency control.

As with the case of real random variables, the distribution of the amplitude and phase for a sample of complex random variables cannot often be expressed in closed form. For instance, while deriving the distribution of the amplitude and phase of a sinusoid in noise, Campbell et al. [2] find that: "The solution is in the form of an integral. Simplifications are possible when the time duration of the process is large and when the signal-to-noise-ratio is very large or very small. Otherwise, complicated numerical integration is required". While studying stochastically perturbed resonance, Neu [11] finds that the distribution of the amplitude and phase can be determined only in the limit "where the time scale of resonance is much shorter than the time scale of diffusion". Sums of random variables needed to compute the amplitude and phase do not have closed form densities even for many commonly known univariate distributions. These include the Weibull [5], generalized normal [12], Rayleigh [6], Rice [7], generalized gamma [13], Nakagami [3], Burr [9], and the lognormal [10] distributions.

Hence, there is a need for approximations for the distribution of the amplitude and phase. The aim of this paper is to provide Edgeworth-type expansions for the joint distribution of (normalized versions of) the amplitude and phase of the sample mean. The results obtained apply to distributions with finite cumulants.

The results are organized as follows. In Section 2, we give Edgeworth-type expansions for the distribution of (normalized versions of) the amplitude and phase of sample means of complex random variables. In Sections 3 and 4, we transform these

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results to polar form for both the *central case*, that is, when the population mean is zero, and the *non-central case*. Suitably normalized amplitude of the sample mean and the phase only have non-degenerate limit distributions in the non-central case. In Section 5, we specialize to the case, where we have a sample of products of correlated complex random variables, in particular products of complex normal random variables. The building blocks of the expansions are the bivariate Hermite polynomials given in Appendix A.

We use the following notation: let $\mathcal{R}$ and $\mathcal{C}$ denote the sets of real and complex numbers. Set $\mathrm{i} = \sqrt{-1}$. Let $P_1, \ldots, P_n$ be independent and identically distributed (i.i.d.) random variables in $\mathcal{C}$. Then the sample mean is $n^{-1}S_n$, where

$$S_n = S_{n,R} + \mathrm{i}S_{n,C} = Amp_n \exp(\mathrm{i}\theta_n) = \sum_{j=1}^{n} P_j \quad \text{with } P_j = R_j + \mathrm{i}C_j. \tag{1.1}$$

So, $Amp_n = |S_n|$ is the amplitude of the sum and Amp_n/n is the amplitude of the sample mean. Also θ_n is the phase of the sum and also the phase of the sample mean.

Proposition 1.1. *We use the convention that the phase, $\theta_n = \arg(S_n)$, lies in $[0, 2\pi)$. Sometimes it is convenient to use instead $\theta_n^* = \tan^{-1}(S_{n,C}/S_{n,R})$ in $[-\pi/2, \pi/2)$. They are related by $\theta_n^* = \theta_n - k\pi$, where $k = 0, 1, 2, 3$ for S_n (or θ_n) in quadrants $1, 2, 3, 4$, respectively.*

We denote the joint cumulants of (R_j, C_j) by $\{\kappa_{r,s}\}$. These are generated by

$$\sum_{r,s=0}^{\infty} u^r v^s \kappa_{r,s}/(r!s!) = \log \mathbb{E}\left[\exp(uR_j + vC_j)\right],$$

and are assumed to be finite. So,

$$\boldsymbol{\mu}' = \mathbb{E}(R_j, C_j) = (\kappa_{1,0}, \kappa_{0,1}) \quad \text{and} \quad \mathbf{V} = Cov(R_j, C_j) = \begin{pmatrix} \kappa_{2,0} & \kappa_{1,1} \\ \kappa_{1,1} & \kappa_{0,2} \end{pmatrix}. \tag{1.2}$$

Proposition 1.2. *The results of this paper still hold under the weaker condition that $P_1, \ldots, P_n$ are independent but not necessarily identically distributed. In this case, $\kappa_{r,s}$ must be interpreted as the average of the corresponding cumulants for $P_1, \ldots, P_n$. For example, if each P_j is replaced by $P_j^* = a_j P_j$, where a_j is a non-random real multiplier, then the results of this paper hold for $P_1^*, \ldots, P_n^*$ with $\kappa_{r,s}$ replaced by*

$$\kappa_{r,s}^* = a_{n,r+s}\kappa_{r,s},$$

where

$$a_{n,r} = n^{-1} \sum_{j=1}^{n} a_j^r,$$

assuming that these are bounded as $n \to \infty$ and that $a_{n,2}$ is bounded away from zero. So, if $\mathbf{V} = v\mathbf{I}_2$, then the corresponding covariance matrix for $P_1^*, \ldots, P_n^*$ is $\mathbf{V}^* = v^*\mathbf{I}_2$, where $v^* = a_{n,2}v$.

Set

$$\mathbf{Y}_n = n^{-1/2} \begin{pmatrix} S_{n,R} - n\kappa_{1,0} \\ S_{n,C} - n\kappa_{0,1} \end{pmatrix}. \tag{1.3}$$

By the Central Limit Theorem,

$$\mathbf{Y}_n \xrightarrow{d} \mathbf{Y} \tag{1.4}$$

as $n \to \infty$, where $\mathbf{Y} \sim \mathcal{N}_2(\mathbf{0}, \mathbf{V})$ in $\mathcal{R}^2$, a bivariate normal with mean $\mathbb{E}[\mathbf{Y}] = \mathbf{0}$ in $\mathcal{R}^2$ and 2×2 covariance matrix $\mathbf{V} = \mathbb{E}[\mathbf{YY}']$. We denote its distribution by $\Phi_{\mathbf{Y}}(\mathbf{y}; \mathbf{V})$ and its density by

$$\phi_{\mathbf{Y}}(\mathbf{y}; \mathbf{V}) = (2\pi)^{-1}|\mathbf{V}|^{-1/2} \exp(-\mathbf{y}'\mathbf{V}^{-1}\mathbf{y}/2)$$

for $\mathbf{y}$ in $\mathcal{R}^2$, assuming $\mathbf{V}$ is positive-definite. The convention in (1.4) for a random variable tending to a random variable is convergence in distribution.

Our main results may be summarized as follows. We distinguish between *the central case*, when $\mathbb{E}[P_j] = 0$ and *the non-central case*, when $\mathbb{E}[P_j] \neq 0$.

The random vector $(Amp_n^2/n, \theta_n)$ has a non-degenerate limit when $\mathbb{E}[P_j] = 0$, but converges to $(\infty, \arg(\mathbb{E}[P_j]))$ when $\mathbb{E}[P_j] \neq 0$. In fact, the amplitude Amp_n of the sum S_n has magnitude $n^{1/2}$ in the central case and magnitude n in the non-central case. Also

$$Amp_n^2/n^2 \xrightarrow{p} |\boldsymbol{\mu}|^2 = |\mathbb{E}[P_j]|^2 = \kappa_{1,0}^2 + \kappa_{0,1}^2.$$