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Hazard rate comparison of parallel systems with heterogeneous gamma components [☆]

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ABSTRACT

We compare the hazard rate functions of the largest order statistic arising from independent heterogeneous gamma random variables and that arising from i.i.d. gamma random variables. Specifically, let $X_1, \dots, X_n$ be independent gamma random variables with X_i having shape parameter $0 < r \le 1$ and scale parameter λ_i, $i = 1, \dots, n$. Denote by $Y_{n:n}$ the largest order statistic arising from i.i.d. gamma random variables $Y_1, \dots, Y_n$ with Y_i having shape parameter r and scale parameter $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of λ_i's. It is shown that $X_{n:n}$ is stochastically larger than $Y_{n:n}$ in terms of hazard rate order. The result derived here strengthens and generalizes some of the results known in the literature and leads to a sharp upper bound on the hazard rate function of the largest order statistic from heterogeneous gamma variables in terms of that of the largest order statistic from i.i.d. gamma variables. A numerical example is finally provided to illustrate the main result established here.

1. Introduction

Order statistics play a prominent role in statistical inference, reliability theory, life testing, operations research, and many other areas; see, for example, the two encyclopedic volumes by [2,3]. Denote by $X_{1:n} \le \cdots \le X_{n:n}$ the order statistics arising from random variables $X_1, \dots, X_n$. Then, it is well-known that the kth order statistic $X_{k:n}$ corresponds to the lifetime of a $(n-k+1)$-out-of-n system, a very popular structure of redundancy in fault-tolerant systems in reliability theory that has been studied extensively in the literature. Series and parallel systems are the building blocks of more complex coherent systems, wherein the lifetime of a parallel system corresponds to the largest order statistic $X_{n:n}$ and the lifetime of a series system corresponds to the smallest order statistic $X_{1:n}$. Many authors have studied various aspects of order statistics when the observations are independent and identically distributed (i.i.d.). The case when observations are non-i.i.d., however, often arises naturally in different situations. Due to the complexity of the distribution theory in this case, limited work can be found in the literature; see, for example, [6,2,3], and the recent review article of Balakrishnan [1] for comprehensive discussions on the independent and non-identically distributed (i.ni.d.) case.

The exponential distribution has a nice mathematical form and the unique memoryless property and hence has widely been applied in many fields. Many papers have appeared on the stochastic comparison of order statistics arising from

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i.ni.d. exponential random variables including [19,20,12,7,9–11,5,13,14,17,18,22–27]. The gamma distribution has been used extensively in reliability and survival analysis due to its flexibility in shape and some nice distributional properties; for more details on this distribution, one may refer to Johnson et al. [8]. Assuming that X is a gamma random variable with shape parameter r and scale parameter λ, X has its probability density function as

$$f(x;r,\lambda)=\frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x},\quad x>0,\ r>0,\ \lambda>0.$$

It is an extremely flexible family of distributions with decreasing, constant, and increasing hazard rates when $0<r<1$, $r=1$ and $r>1$, respectively. This paper will focus on the largest order statistic arising from heterogeneous gamma variables, i.e., the lifetime of a parallel system with independent heterogeneous gamma components. The results established here extend the corresponding ones in the literature for the exponential case.

Let us first recall some notions of stochastic orders. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*. For two random variables X and Y with densities f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\overline{F}_X=1-F_X$ and $\overline{F}_Y=1-F_Y$ be the corresponding survival functions. X is said to be smaller than Y in the likelihood ratio order (denoted by $X\le_{\mathrm{lr}}Y$) if $f_Y(x)/f_X(x)$ is increasing in x; X is said to be smaller than Y in the hazard rate order (denoted by $X\le_{\mathrm{hr}}Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in x; X is said to be smaller than Y in the stochastic order (denoted by $X\le_{\mathrm{st}}Y$) if $\overline{F}_Y(x)\ge\overline{F}_X(x)$. It is well-known that the likelihood ratio order implies the hazard rate order which in turn implies the usual stochastic order. For a comprehensive discussion on various stochastic orderings, one may refer to Shaked and Shanthikumar [21].

Let $X_1,\dots,X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i=1,\dots,n$. Let $Y_1,\dots,Y_n$ be a random sample of size n from an exponential distribution with hazard rate $\bar{\lambda}=\sum_{i=1}^n\lambda_i/n$, the arithmetic mean of $\lambda_i's$, and denote by $Y_{n:n}$ the corresponding largest order statistic. Dykstra et al. [7] then showed that

$$X_{n:n}\ge_{\mathrm{hr}}Y_{n:n},\tag{1}$$

which was further strengthened by Kochar and Xu [13] as

$$X_{n:n}\ge_{\mathrm{lr}}Y_{n:n}.\tag{2}$$

[10] also strengthened the result in (1), under a weaker condition, by proving that if $Z_1,\dots,Z_n$ is a random sample of size n from an exponential distribution with hazard rate $\tilde{\lambda}=\left(\prod_{i=1}^n\lambda_i\right)^{\frac{1}{n}}$, the geometric mean of $\lambda_i's$, then

$$X_{n:n}\ge_{\mathrm{hr}}Z_{n:n}.\tag{3}$$

Recently, Kochar and Xu [14] proved that the largest order statistic from heterogeneous exponential variables is more skewed in the sense of the convex transform order than that from homogeneous exponential variables, which is quite a general conclusion as there is no restriction on the parameters.

It is natural to ask whether and how the result in (3) can be extended from the exponential case to the gamma distribution. This paper confirms this result for the case when the shape parameter is at most 1. Specifically, let $X_1,\dots,X_n$ be independent gamma random variables with X_i having shape parameter $0<r\le 1$ and scale parameter λ_i, $i=1,\dots,n$, and let $Z_1,\dots,Z_n$ be a random sample of size n from a gamma distribution with shape parameter r and scale parameter $\tilde{\lambda}=\left(\prod_{i=1}^n\lambda_i\right)^{\frac{1}{n}}$. We then show that

$$X_{n:n}\ge_{\mathrm{hr}}Z_{n:n},\tag{4}$$

thus generalizing and strengthening the corresponding result for the exponential case established earlier in the literature.

2. Main result

In this section, before presenting our main result, we first present several useful lemmas. The first one, due to Bon and Păltănea [5], plays an important role in establishing the main result which presents a sufficient condition to reach the maximum point of a symmetrical function in a compact set.

Lemma 1. *Let $\phi:\Re_+^n\to\Re_+$ be a symmetric and continuously differentiable mapping. If for any n-dimensional vector $\mathbf{y}=(y_1,\dots,y_n)\in\Re_+^n$ with $y_p=\min y_i$ and $y_q=\max y_i$, we have*

$$(y_p-y_q)\left(\frac{\frac{\partial\phi(\mathbf{y})}{\partial y_p}}{\sum_{i\neq p}^n y_i}-\frac{\frac{\partial\phi(\mathbf{y})}{\partial y_q}}{\sum_{i\neq q}^n y_i}\right)<0,\quad \textit{for } y_p\neq y_q,$$

then the following inequality holds:

$$\phi(y_1,\dots,y_n)\le\phi(\underbrace{\tilde{y},\dots,\tilde{y}}_{n}),$$