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An exact test about the covariance matrix

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ABSTRACT

In the present paper, we propose an exact test on the structure of the covariance matrix. In its development the properties of the Wishart distribution are used. Unlike the classical likelihood-ratio type tests and the tests based on the empirical distance, whose statistics depend on the total variance and the generalized variance only, the proposed approach provides more information about the changes in the covariance matrix. Via an extensive simulation study the new approach is compared with the existent asymptotic tests.

1. Introduction

Tests about the covariance matrix have significantly increased its popularity recently. Historically, the first test on the covariance matrix was suggested by Mauchly [16] that is based on the likelihood ratio approach. Because the statistic of this test depends on the determinant and the trace of the sample covariance matrix, the so-called generalized and total variances respectively, it requires that the sample covariance matrix is non-singular which is the case with probability one when the sample size is larger than the process dimension. Gupta and Xu [8] extended the likelihood-ratio test to non-normal distributions by deriving the asymptotic expansion of the test statistic under the null hypothesis, while Bai et al. [2] considered a modification of the likelihood-ratio test. The second approach considered in the statistical literature is based on the empirical distance initially suggested by John [11] and Nagao [18]. These test statistics with some modifications can also be applied for testing the covariance matrix in case of high-dimensional data (cf. [14,6]) even when the sample size is smaller than the process dimension. Other approaches are based on the largest eigenvalue of the covariance matrix [12,13] or they are derived by using the methods of random matrix theory (cf. [5]).

In this paper we derive an exact test based on the examination of a fixed column of the sample covariance matrix. In the development of this test the properties of the Wishart distribution are used. Since an exact test is developed it is always correctly sized. Moreover, the suggested test can also be applied if the sample size is much smaller than the dimension of the process. Via an extensive simulation study we show that the new approach performs very well if changes in a few elements of the covariance matrix take place.

The rest of the paper is structured as follows. In Section 2, we introduce a test about the covariance matrix. The distribution of the test statistic is derived under both the null and alternative hypothesis. In Section 3, an extension of the test

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is provided. A very useful stochastic representation of the test statistic is obtained under H_0 which shows that under the null hypothesis the distribution is independent of the target matrix specified under H_0. In Section 4, the distributional properties of the test statistic under H_1 are studied via an extensive Monte-Carlo study. Some proofs are given in the Appendix.

2. Test based on a column of the covariance matrix

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \sim ii\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}),\ p > 2$, be an independent sample from the multivariate normal distribution with known mean vector $\boldsymbol{\mu}$. Without loss of generality we assume that $\boldsymbol{\mu} = \mathbf{0}_p$, where $\mathbf{0}_p$ stands for the p-dimensional vector of zeros. The covariance matrix $\boldsymbol{\Sigma}$ is estimated by

$$\mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'. \tag{1}$$

If $\boldsymbol{\mu}$ is an unknown quantity then instead of (1) we use the sample covariance matrix for estimating $\boldsymbol{\Sigma}$ expressed as

$$\tilde{\mathbf{S}} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})' \quad \text{with} \quad \bar{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i. \tag{2}$$

The two approaches differ only slightly from each other since it holds that (see, e.g. [17, p. 90])

$$n\mathbf{S} \sim W_p(n, \boldsymbol{\Sigma}) \quad \text{and} \quad (n-1)\tilde{\mathbf{S}} \sim W_p(n-1, \boldsymbol{\Sigma}),$$

where the symbol $W_p(n, \boldsymbol{\Sigma})$ stands for the p-dimensional Wishart distribution with n degrees of freedom and covariance matrix $\boldsymbol{\Sigma}$ (cf. [17,20]). Moreover, both estimators are unbiased as well as asymptotically normally distributed [17, pp. 90–91]. The last result is usually used for the derivation of asymptotic tests on the covariance matrix.

In this paper we consider an alternative approach that is based on the distributional properties of the Wishart distribution and the singular Wishart distribution [20,4]. First, an exact test is proposed which is based on a column of the sample covariance matrix and then it is generalized. In the derivation, no assumption on p, like $n \geq p$, is imposed. The results hold in all possible cases, i.e. for $n \geq p$ and $n < p$. While the properties of the Wishart distribution are applied for $n \geq p$, we make use of the distributional results derived for the singular Wishart distribution in the case of $n < p$.

We assume that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ under H_0 and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$ under H_1. The matrices $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_1$, and $\mathbf{S}$ are partitioned as follows

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} \xi_0 & \boldsymbol{\nu}_0' \\ \boldsymbol{\nu}_0 & \boldsymbol{\Xi}_0 \end{bmatrix}, \qquad \boldsymbol{\Sigma}_1 = \begin{bmatrix} \xi_1 & \boldsymbol{\nu}_1' \\ \boldsymbol{\nu}_1 & \boldsymbol{\Xi}_1 \end{bmatrix}, \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} v & \mathbf{t}' \\ \mathbf{t} & \mathbf{W} \end{bmatrix}. \tag{3}$$

Let $\boldsymbol{\Upsilon}_0 = \boldsymbol{\Xi}_0 - \boldsymbol{\nu}_0\boldsymbol{\nu}_0'/\xi_0$ and $\boldsymbol{\Upsilon}_1 = \boldsymbol{\Xi}_1 - \boldsymbol{\nu}_1\boldsymbol{\nu}_1'/\xi_1$. Without loss of generality we now present a test based on the first column of the covariance matrix $\boldsymbol{\Sigma}_0$. In case of the ith column the test statistic can be derived similarly. Here, instead of the partitions (3), we construct the partition for the (i,i)-th element of the matrices $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_1$, and $\mathbf{S}$ as follows. Let $\xi_{0,i}$ denote the (i,i)-th element of the matrix $\boldsymbol{\Sigma}_0$, $i = 1, \ldots, p$. By $\boldsymbol{\nu}_{0,i}$ we denote the ith column of the matrix $\boldsymbol{\Sigma}_0$ without $\xi_{0,i}$. Let $\boldsymbol{\Xi}_{0,i}$ denote a square matrix of order $p-1$, which is obtained from the matrix $\boldsymbol{\Sigma}_0$ by deleting the ith row and the ith column. Finally, $\boldsymbol{\Upsilon}_{0,i} = \boldsymbol{\Xi}_{0,i} - \boldsymbol{\nu}_{0,i}\boldsymbol{\nu}_{0,i}'/\xi_{0,i}$ is calculated. In the same way we define $\xi_{1,i},\ \boldsymbol{\nu}_{1,i},\ \boldsymbol{\Xi}_{1,i},\ \boldsymbol{\Upsilon}_{1,i},\ v_i,\ \mathbf{t}_i$, and $\mathbf{W}_i$ by splitting $\boldsymbol{\Sigma}_1$ and $\mathbf{S}$ correspondingly. For presentation purposes we drop the index i in the notations if $i = 1$.

The hypotheses are given by

$$H_0 : \boldsymbol{\Sigma} = d\boldsymbol{\Sigma}_0 \quad \text{against} \quad H_1 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1 \neq d\boldsymbol{\Sigma}_0, \tag{4}$$

where $d > 0$ denotes an arbitrary (un)known constant. We define

$$\boldsymbol{\eta}_1 = \sqrt{n}\boldsymbol{\Upsilon}_0^{-1/2}\left(\frac{\mathbf{t}}{v} - \frac{\boldsymbol{\nu}_0}{\xi_0}\right)v^{1/2}. \tag{5}$$

Let $\phi_k(.; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ stand for the density function of the k-dimensional multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. In Theorem 1 we derive the distributions of the random vector $\boldsymbol{\eta}_1$ under both H_0 and H_1 hypotheses.

Theorem 1. (a) *Let $\mathbf{X}_i \sim iid\mathcal{N}_p(\mathbf{0}_p, \boldsymbol{\Sigma}_1),\ i = 1, \ldots, n$. Then the density function of $\boldsymbol{\eta}_1$ is given by*

$$f_{\boldsymbol{\eta}_1}(\mathbf{x}) = 2\frac{\sqrt{\pi}\,n^{(n-1)/2}}{2^{(n-1)/2}\xi_1^{(n-1)/2}\Gamma\left(\frac{n}{2}\right)}\phi_{p-1}\left(\mathbf{x}; \mathbf{0}_{p-1}, \boldsymbol{\Omega} + \xi_1\boldsymbol{\Delta}\boldsymbol{\Delta}'\right) \times \int_0^\infty y^{n-1}\phi_1\left(y; \frac{\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\mathbf{x}}{\sqrt{n}(\xi_1^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})}, n^{-1}(\xi_1^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})^{-1}\right)dy, \tag{6}$$

where

$$\boldsymbol{\Delta} = \boldsymbol{\Upsilon}_0^{-1/2}\left(\frac{\boldsymbol{\nu}_1}{\xi_1} - \frac{\boldsymbol{\nu}_0}{\xi_0}\right) \quad \text{and} \quad \boldsymbol{\Omega} = \boldsymbol{\Upsilon}_0^{-1/2}\boldsymbol{\Upsilon}_1\boldsymbol{\Upsilon}_0^{-1/2}.$$

(b) *Let $\mathbf{X}_i \sim iid\mathcal{N}_p(\mathbf{0}_p, \boldsymbol{\Sigma}_0),\ i = 1, \ldots, n$. Then $\boldsymbol{\eta}_1 \sim \mathcal{N}_{p-1}(\mathbf{0}_{p-1}, \mathbf{I}_{p-1})$, where $\mathbf{I}_k$ denotes a $k \times k$ identity matrix.*