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A generalized multivariate kurtosis ordering and its applications

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ABSTRACT

It has been commonly admitted that the meaning of a descriptive feature of distributions is given by an ordering and that the measures for this feature are meaningful only if they preserve the ordering. However, while many multivariate kurtosis measures have been introduced, multivariate kurtosis orderings have received relatively little investigation. In this paper, we propose and study a generalized multivariate kurtosis ordering. Under some conditions, this ordering is affine invariant and determines elliptically symmetric distributions within affine equivalence. Some special cases of the generalized ordering provide the kurtosis orderings for various existing multivariate kurtosis measures. Those kurtosis orderings are applied to explore the relationships of the multivariate kurtosis measures. Some other applications of the generalized multivariate kurtosis ordering are also given.

1. Introduction

Unlike location, spread and skewness, the meaning of kurtosis is a topic of considerable debate. Even with regard to the basic question of what kurtosis measures, there is no universal agreement up to now. Thus various multivariate kurtosis measures have been proposed.

The classical notion of univariate kurtosis is moment-based and given by the standardized fourth central moment $\beta_1(F) = E(X-\mu)^4/\sigma^4$, where F is the distribution function of X. As a measure of a key descriptive feature of univariate distributions, $\beta_1(F)$ has broad applications, for example, test for univariate normality, detection of univariate outliers, risk analysis (kurtosis risk), image sharpness, and so on. A natural multivariate extension of $\beta_1(F)$ was given by Mardia [9] as the fourth moment of the Mahalanobis distance of a random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ from its mean $\boldsymbol{\mu}$, i.e.,

$$\beta_d(F) = E[(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})]^2.$$

Srivastava [14] generalized $\beta_1(F)$ to $\beta_d^*(F)$, defined as the average of kurtosis values of the principal components for the multivariate case. Since principal components are related to direction, $\beta_d^*(F)$ is not affine invariant. Utilizing simplicial volume, Oja [10] proposed another multivariate extension of $\beta_1(F)$. Besides those moment-based multivariate kurtosis measures, various nonparametric multivariate kurtosis measures have also been proposed.

Treating kurtosis and tailweight as the same notion, Liu et al. [6] introduced a depth-based multivariate kurtosis measure, a "fan plot", exhibiting several curves of

$$b_F(t|p_0) = \frac{V_F(tp_0)}{V_F(p_0)}, \quad 0 \le t \le 1,$$

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for selected choices of p_0, where $V_F(p)$ denotes the depth-based volume functional (see Section 2 for details). They also introduced other forms of depth-based multivariate kurtosis measures, i.e., a Lorenz curve, and a "shrinkage plot". A general depth-based Lorenz curve was discussed by Serfling [11]. Extending the Groeneveld and Meeden [4] kurtosis measure for univariate symmetric distributions, Wang and Serfling [18] introduced a nonparametric multivariate kurtosis functional:

$$k_F(p) = \frac{V_F\left(\frac{1}{2}+\frac{p}{2}\right) + V_F\left(\frac{1}{2}-\frac{p}{2}\right) - 2V_F\left(\frac{1}{2}\right)}{V_F\left(\frac{1}{2}+\frac{p}{2}\right) - V_F\left(\frac{1}{2}-\frac{p}{2}\right)}, \quad 0 < p < 1.$$

Other multivariate kurtosis measures can be found in Malkovich and Afifi [8], Averous and Meste [1], and Serfling [12].

It can be seen that the above multivariate kurtosis measures are quite different and, of course, for a distribution F in $\mathbb{R}^d$ they give quite different kurtosis values in general. Here the questions of interest are the following.

(1) Is there any relationship among those multivariate kurtosis measures? Especially are they consistent (that is, for two distributions F and G in $\mathbb{R}^d$, if the kurtosis of G is higher than the one of F according to a kurtosis measure, does the same kurtosis relationship hold according to another kurtosis measure)?

(2) For the "fan plot", is it necessary to use different choices of p_0?

(3) For the nonparametric multivariate kurtosis functional $k_F(p)$, can we use the value of $k_F(p)$ at a single point p instead of the whole curve?

Such questions motivate this work. To answer those questions, we study multivariate kurtosis by the ordering approach. A generalized multivariate kurtosis ordering is proposed and studied in Section 2. Section 3 focuses on some special cases of the ordering, which provide the kurtosis orderings for various existing multivariate kurtosis measures. Those kurtosis orderings are applied to explore the relationships of various multivariate kurtosis measures in Section 4. Section 5 is devoted to some other applications of the generalized multivariate kurtosis ordering.

Throughout this paper, we confine attention to continuous distributions. We use uppercase letters to denote distribution functions and their lowercase counterparts to denote density functions. For example, we denote by $F_{\boldsymbol{X}}$ and $f_{\boldsymbol{X}}$ the cdf and density of a random vector $\boldsymbol{X}$ in $\mathbb{R}^d$. When X is a random variable, the quantile function of X is denoted by F_X^{-1}. Without confusion, we will omit the subscript.

2. A generalized multivariate kurtosis ordering

Extending the van Zwet [16] kurtosis ordering for univariate symmetric distributions, Balanda and MacGillivray [2] proposed a univariate kurtosis ordering, which involves the case of univariate asymmetric distributions: $F \leqslant_s G$ if and only if (iff) $S_G(S_F^{-1}(r))$ is convex for $r \geq 0$, equivalently, $S_F(p) \leq_c S_G(p)$, where $\leq_c$ is the van Zwet [16] skewness ordering for univariate distributions, $S_F(p)$ and $S_G(p)$ are the spread functions of F and G, respectively, i.e.,

$$S_F(p) = F^{-1}\left(\frac{1}{2}+\frac{p}{2}\right) - F^{-1}\left(\frac{1}{2}-\frac{p}{2}\right), \qquad S_G(p) = G^{-1}\left(\frac{1}{2}+\frac{p}{2}\right) - G^{-1}\left(\frac{1}{2}-\frac{p}{2}\right).$$

Here comes out a general approach to develop a multivariate kurtosis ordering. Given a spread functional of distributions in $\mathbb{R}^d$, a univariate skewness ordering on the spread functions will yield a kurtosis ordering for the underlying distributions. See Averous and Meste [1] and Wang [17] for detailed discussion. In this section, we will develop a generalized multivariate kurtosis ordering by a generalized depth-based spread functional.

2.1. Definition

In recent years, statistical depth functions are playing an increasingly important role in nonparametric multivariate analysis. Generally, a depth function $D_F(\boldsymbol{x})$ is a nonnegative real-valued mapping which provides a distribution-based center-outward ordering of points $\boldsymbol{x}$ in $\mathbb{R}^d$. Given a depth function, the center of the distribution is defined as the point of maximal depth, a multidimensional median, and in typical cases it agrees with the center as defined by a notion of symmetry. Desirable properties for a depth function are: (1) affine invariance ($D_{F_{\mathbf{A}\boldsymbol{X}+\boldsymbol{b}}}(\mathbf{A}\boldsymbol{x}+\boldsymbol{b}) = D_{F_{\boldsymbol{X}}}(\boldsymbol{x})$ for any nonsingular $d \times d$ matrix $\mathbf{A}$ and d-vector $\boldsymbol{b}$); (2) maximality at "center"; (3) monotonicity relative to the deepest point; (4) vanishing at infinity. The following are some widely used depth functions.

The Mahalanobis depth. The Mahalanobis depth is defined by the Mahalanobis [7] distance as

$$MD_F(\boldsymbol{x}) = \left[1 + (\boldsymbol{x} - \boldsymbol{\mu}_F)' \boldsymbol{\Sigma}_F^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_F)\right]^{-1}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where $\boldsymbol{\mu}_F$ and $\boldsymbol{\Sigma}_F$ are the mean vector and covariance matrix of F, respectively.

The halfspace depth. Tukey [15] introduced the halfspace depth,

$$HD_F(\boldsymbol{x}) = \inf\{P(H) : \boldsymbol{x} \in H \in \mathcal{H}\}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where $\mathcal{H} = \{\text{all closed halfspaces}\}$.