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Bivariate gamma-geometric law and its induced Lévy process

Wagner Barreto-Souza

Departamento de Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo/SP 05508-090, Brazil

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ABSTRACT

In this article we introduce a three-parameter extension of the bivariate exponential-geometric (BEG) law (Kozubowski and Panorska, 2005) [4]. We refer to this new distribution as the bivariate gamma-geometric (BGG) law. A bivariate random vector (X, N) follows the BGG law if N has geometric distribution and X may be represented (in law) as a sum of N independent and identically distributed gamma variables, where these variables are independent of N. Statistical properties such as moment generation and characteristic functions, moments and a variance–covariance matrix are provided. The marginal and conditional laws are also studied. We show that BBG distribution is infinitely divisible, just as the BEG model is. Further, we provide alternative representations for the BGG distribution and show that it enjoys a geometric stability property. Maximum likelihood estimation and inference are discussed and a reparametrization is proposed in order to obtain orthogonality of the parameters. We present an application to a real data set where our model provides a better fit than the BEG model. Our bivariate distribution induces a bivariate Lévy process with correlated gamma and negative binomial processes, which extends the bivariate Lévy motion proposed by Kozubowski et al. (2008) [6]. The marginals of our Lévy motion are a mixture of gamma and negative binomial processes and we named it BMixGNB motion. Basic properties such as stochastic self-similarity and the covariance matrix of the process are presented. The bivariate distribution at fixed time of our BMixGNB process is also studied and some results are derived, including a discussion about maximum likelihood estimation and inference.

1. Introduction

Mixed univariate distributions have been introduced and studied in the last few years by compounding continuous and discrete distributions. Marshall and Olkin [10] introduced a class of distributions which can be obtained by the minimum and maximum of independent and identically distributed (iid) continuous random variables (independent of the random sample size), where the sample size follows a geometric distribution.

Chahkandi and Ganjali [2] introduced some lifetime distributions by compounding exponential and power series distributions; these models are called exponential power series (EPS) distributions. Recently, Morais and Barreto-Souza [11] introduced a class of distributions obtained by mixing Weibull and power series distributions and studied several of its statistical properties. This class contains the EPS distributions and other lifetime models studied recently, for example, the Weibull-geometric distribution [10,1]. The reader is referred to the introduction from Morais and Barreto-Souza's [11] article for a brief literature review about some univariate distributions obtained by compounding.

A mixed bivariate law with exponential and geometric marginals was introduced by Kozubowski and Panorska [4], and named the bivariate exponential-geometric (BEG) distribution. A bivariate random vector (X, N) follows the BEG law if it

E-mail addresses: wagnerbs85@gmail.com, wagnerbs@ime.usp.br.

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admits the stochastic representation:

$$(X, N) \stackrel{d}{=} \left(\sum_{i=1}^{N} X_i, N\right), \tag{1}$$

where the variable N follows a geometric distribution and $\{X_i\}_{i=1}^{\infty}$ is a sequence of iid exponential variables, independent of N. The BEG law is infinitely divisible and therefore leads a bivariate Lévy process, in this case, with gamma and negative binomial marginal processes. This bivariate process, named BGNB motion, was introduced and studied by Kozubowski et al. [6].

Other multivariate distributions involving exponential and geometric distributions have been studied in the literature. Kozubowski and Panorska [5] introduced and studied a bivariate distribution involving a geometric maximum of exponential variables. A trivariate distribution involving geometric sums and maximum of exponential variables was also recently introduced by Kozubowski et al. [7].

Our chief goal in this article is to introduce a three-parameter extension of the BEG law. We refer to this new three-parameter distribution as the bivariate gamma-geometric (BGG) law. Further, we show that this extended distribution is infinitely divisible, and, therefore, it induces a bivariate Lévy process which has the BGNB motion as a particular case. The additional parameter controls the shape of the continuous part of our models.

Our bivariate distribution may be applied in areas such as hydrology and finance. We here focus in finance applications and use the BGG law for modeling log-returns (the X_i's) corresponding to a daily exchange rate. More specifically, we are interested in modeling cumulative log-returns (the X) in growth periods of the exchange rates. In this case N represents the duration of the growth period, where the consecutive log-returns are positive. As mentioned by Kozubowski and Panorska [4], the geometric sum represented by X in (1) is very useful in several fields including water resources, climate research and finance. We refer the reader to the introduction from the article by [4] for a good discussion on practical situations where the random vectors with description (1) may be useful.

The present article is organized as follows. In Section 2 we introduce the bivariate gamma-geometric law and derive basic statistical properties, including a study of some properties of its marginal and conditional distributions. Further, we show that our proposed law is infinitely divisible. Estimation by maximum likelihood and inference for large samples are addressed in Section 3, which also contains a proposed reparametrization of the model in order to obtain orthogonality of the parameter in the sense of Cox and Reid [3]. An application to a real data set is presented in Section 4. The induced Lévy process is approached in Section 5 and some of its basic properties are shown. We include a study of the bivariate distribution of the process at fixed time and also discuss estimation of the parameters and inferential aspects. We close the article with concluding remarks in Section 6.

2. The law and basic properties

The bivariate gamma-geometric (BGG) law is defined by the stochastic representation (1) and assuming that $\{X_i\}_{i=1}^{\infty}$ is a sequence of iid gamma variables independent of N and with a probability density function given by $g(x; \alpha, \beta) = \beta^\alpha/\Gamma(\alpha)x^{\alpha-1}e^{-\beta x}$, for $x > 0$ and $\alpha, \beta > 0$; we denote $X_i \sim \Gamma(\alpha, \beta)$. As before, N is a geometric variable with probability mass function given by $P(N = n) = p(1-p)^{n-1}$, for $n \in \mathbb{N}$; denote $N \sim \text{Geom}(p)$. Clearly, the BGG law contains the BEG law as a particular case, for the choice $\alpha = 1$. The joint density function $f_{X,N}(\cdot, \cdot)$ of (X, N) is given by

$$f_{X,N}(x, n) = \frac{\beta^{n\alpha}}{\Gamma(\alpha n)} x^{n\alpha-1} e^{-\beta x} p(1-p)^{n-1}, \quad x > 0,\ n \in \mathbb{N}. \tag{2}$$

Hence, it follows that the joint cumulative distribution function (cdf) of the BGG distribution can be expressed by

$$P(X \le x, N \le n) = p\sum_{j=1}^{n}(1-p)^{j-1}\frac{\Gamma_{\beta x}(j\alpha)}{\Gamma(j\alpha)},$$

for $x > 0$ and $n \in \mathbb{N}$, where $\Gamma_x(\alpha) = \int_0^x t^{\alpha-1}e^{-t}dt$ is the incomplete gamma function. We will denote $(X, N) \sim \text{BGG}(\beta, \alpha, p)$. We now show that $(pX, pN) \stackrel{d}{\to} (\alpha Z/\beta, Z)$ as $p \to 0^+$, where '$\stackrel{d}{\to}$' denotes convergence in distribution and Z is a exponential variable with mean 1; for $\alpha = 1$, we obtain the result given in proposition 2.3 from [4]. For this, we use the moment generation function of the BGG distribution, which is given in Section 2.2. Hence, we have that $E(e^{tpX+spN}) = \varphi(pt, ps)$, where $\varphi(\cdot, \cdot)$ is given by (4). Using L'Hôpital's rule, one may check that $E(e^{tpX+spN}) \to (1 - s - \alpha t/\beta)^{-1}$ as $p \to 0^+$, which is the moment generation function of $(\alpha Z/\beta, Z)$.

2.1. Marginal and conditional distributions

The marginal density of X with respect to the Lebesgue measure is an infinite mixture of gamma densities, which is given by

$$f_X(x) = \sum_{n=1}^{\infty} P(N = n)g(x; n\alpha, \beta) = \frac{px^{-1}e^{-\beta x}}{1-p}\sum_{n=1}^{\infty}\frac{[(\beta x)^\alpha(1-p)]^n}{\Gamma(n\alpha)}, \quad x > 0. \tag{3}$$