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Moments of MGOU processes and positive semidefinite matrix processes

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ABSTRACT

Moment conditions for multivariate generalized Ornstein–Uhlenbeck (MGOU) processes are derived and the first and second moments are given in terms of the driving Lévy processes. In the second part of the paper a class of multivariate, positive semidefinite processes of MGOU-type is developed and suggested for use as squared volatility process in multivariate financial modeling.

1. Introduction

For any starting random variable $V_0 \in \mathbb{R}^{d\times n}$ the *multivariate generalized Ornstein–Uhlenbeck (MGOU) process* $(V_t)_{t\geq 0}$, $V_t \in \mathbb{R}^{d\times n}$, has been defined in [5] by

$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}\, dY_s\right) \tag{1.1}$$

for the driving Lévy process $(X_t, Y_t)_{t\geq 0}$ with $(X_t, Y_t) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times n}$ such that

$$\det(I + \Delta X_s) \neq 0, \tag{1.2}$$

which guarantees $\det(\overleftarrow{\mathcal{E}}(X)_t) \neq 0$ for all $t \geq 0$.

Hereby for a semimartingale $(X_t)_{t\geq 0}$ in $\mathbb{R}^{d\times d}$ its so called *left stochastic exponential* $\overleftarrow{\mathcal{E}}(X)_t$ is defined as the unique $\mathbb{R}^{d\times d}$-valued, adapted, càdlàg solution $(Z_t)_{t\geq 0}$ of the SDE

$$Z_t = I + \int_{(0,t]} Z_{s-}\, dX_s, \quad t \geq 0, \tag{1.3}$$

while the unique adapted, càdlàg solution $(Z_t)_{t\geq 0}$ of the SDE

$$Z_t = I + \int_{(0,t]} dX_s\, Z_{s-}, \quad t \geq 0, \tag{1.4}$$

is called *right stochastic exponential* and denoted by $\overrightarrow{\mathcal{E}}(X)_t$.

It has been shown in [5] that, under some natural conditions, the MGOU process is the only continuous-time càdlàg process which fulfills for all $h > 0$ a random recurrence equation of the form $V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh}$ for random functionals $(A_{(n-1)h,nh}, B_{(n-1)h,nh}) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times n}$ such that $(A_{(n-1)h,nh}, B_{(n-1)h,nh})_{n\in\mathbb{N}}$ are i.i.d. distributed and $A_{(n-1)h,nh}$ is non-singular for all $h > 0$. Conversely one can see directly from (1.1) that the MGOU process V_t fulfills

$$V_t = A_{s,t}V_s + B_{s,t}, \quad 0 \leq s \leq t, \tag{1.5}$$

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for

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} := \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \end{pmatrix}, \quad 0 \le s \le t. \tag{1.6}$$

It has also been shown in [5] that the MGOU process is the unique solution of the stochastic differential equation

$$dV_t = dU_t V_{t-} + dL_t \tag{1.7}$$

for the Lévy process $(U_t, L_t)_{t\ge 0}$ in $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times n}$ given by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} := \begin{pmatrix} -X_t + [X,X]_t^c + \sum_{0<s\le t} \left((I+\Delta X_s)^{-1} - I + \Delta X_s \right) \\ Y_t + \sum_{0<s\le t} \left((I+\Delta X_s)^{-1} - I \right) \Delta Y_s - [X,Y]_t^c \end{pmatrix}, \quad t \ge 0 \tag{1.8}$$

where the relation between U and X is equivalent to stating $\overrightarrow{\mathcal{E}}(U)_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}$.

We refer to [5] for more details and specific examples of MGOU processes.

As already remarked in [5], MGOU processes have a wide range of possible applications, as they represent on the one hand a multidimensional generalization of generalized Ornstein–Uhlenbeck (GOU) processes which are common to use as volatility models but also appear in storage theory and risk theory (see e.g. [2,12,11] to name just a few), and on the other hand MGOU processes are the continuous time analogon of multidimensional random recurrence equations, which are widely used models in finance, biology and other fields.

To pursue the way of fitting MGOU processes to possible applications in this paper we will first investigate moment conditions and develop the first and second moments of stationary MGOU processes. The results will be given in Section 3 while their rather technical proofs are postponed to Section 5.

In Section 4 we will then consider a way to construct positive semidefinite multivariate processes which are strongly related to MGOU processes. The motivation for this section comes from the fact that when using (one-dimensional) GOU processes as volatility models, the volatility process is usually described as the square-root process of a GOU process. To be able to define a uniquely determined square-root process of a matrix valued process we thus need to determine conditions under which the developed processes only take values in $\mathbb{S}_d^+$, the cone of positive semidefinite matrices in $\mathbb{R}^{d\times d}$.

In his thesis [16] (also see [3,14]) Stelzer has already obtained various results on matrix valued, positive semidefinite, so called Ornstein–Uhlenbeck-type processes. For example [16, Theorems 4.4.5 and 6.2.1], he shows that if A is some matrix with real parts of all eigenvalues strictly negative, then the differential equation

$$dW_t = (AW_{t-} + W_{t-}A^T)\, dt + dL_t$$

has a unique strictly stationary solution given by

$$W_t = e^{At} W_0 e^{A^T t} + \int_{(0,t]} e^{A(t-s)} dL_s e^{A^T(t-s)} = \int_{-\infty}^{t} e^{A(t-s)} dL_s e^{A^T(t-s)} \tag{1.9}$$

and he defines and examines properties of the square-root process of W.

In Section 4 we will introduce the MGOU-type process

$$W_t = \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(W_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}\, dY_s\, (\overleftarrow{\mathcal{E}}(X)_{s-})^T \right) (\overleftarrow{\mathcal{E}}(X)_t^{-1})^T, \tag{1.10}$$

driven by some $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$ valued Lévy process $(X_t, Y_t)_{t\ge 0}$. This process includes (1.9) as a special case and we will show that the corresponding vectorized process $\mathrm{vec}(W)$ is a MGOU process. This allows us to apply the results on MGOU processes derived in [5] and in this paper. In particular we develop the stochastic differential equation of W as given in (4.10) and give moment conditions as well as the first and second moments of W in terms of the driving Lévy process. Finally, in Theorem 4.8 we prove that W is a positive semidefinite process whenever Y is a matrix subordinator, i.e. only has positive semidefinite increments.

2. Preliminaries and notation

Throughout this paper for any matrix $M \in \mathbb{R}^{d\times n}$ we write M^T for its transpose and let $M^{(i,j)}$ denote the component in the ith row and jth column of M. By $\mathrm{vec}(\cdot)$ we denote the vectorization operator which maps any matrix in $\mathbb{R}^{d\times n}$ to the vector in $\mathbb{R}^{dn}$ by stacking its columns one under another. Using vec^{-1} we regain the matrix M from $\mathrm{vec}(M)$. The identity matrix will be written as I. The symbol $\otimes$ denotes the Kronecker product. Norms of vectors and matrices are denoted by $\|\cdot\|$. If the norm is not specified it is irrelevant which specific norm is used but we will always assume it to be submultiplicative. The