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Efficient algorithm for estimating the parameters of a chirp signal

Ananya Lahiri, Debasis Kundu *, Amit Mitra

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India

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ABSTRACT

Chirp signals play an important role in the statistical signal processing. Recently Kundu and Nandi (2008) [8] derived the asymptotic properties of the least squares estimators of the unknown parameters of the chirp signals model in the presence of stationary noise. Unfortunately they did not discuss any estimation procedures. In this article we propose a computationally efficient algorithm for estimating different parameters of a chirp signal in presence of stationary noise. From proper initial guesses, the proposed algorithm produces efficient estimators in a fixed number of iterations. We also suggest how to obtain the proper initial guesses. The proposed estimators are consistent and asymptotically equivalent to least squares estimators of the corresponding parameters. We perform some simulation experiments to see the effectiveness of the proposed method, and it is observed that the proposed estimators perform very well. For illustrative purposes, we have performed the data analysis of a simulated data set. Finally, we propose some generalization in the conclusions.

1. Introduction

In this paper we consider the following chirp signal model in the presence of additive noise:

$$y(n) = A_0 \cos(\alpha_0 n + \beta_0 n^2) + B_0 \sin(\alpha_0 n + \beta_0 n^2) + X(n); \quad n = 1, \dots, N, \tag{1}$$

where A_0, B_0 are non zero amplitudes, with restriction $A_0^2 + B_0^2 \le M$, for some constant M. The frequency and frequency rate, α_0, β_0, respectively, lie strictly between 0 and π. $X(n)$ is a stationary noise sequence, and it has the following form;

$$X(n) = \sum_{j=-\infty}^{\infty} a(j)\varepsilon(n-j), \qquad \sum_{j=-\infty}^{\infty} |a(j)| < \infty. \tag{2}$$

Here, $\{\varepsilon(n)\}$ is a sequence of independent and identically distributed (*i.i.d.*) random variables with zero mean, variance σ^2 and with finite fourth moment. Given a sample of size N, the problem is to estimate the unknown amplitudes A_0, B_0, the frequency α_0 and the frequency rate β_0.

A chirp signal occurs quite naturally in different areas of science and technology. It is a signal where frequency changes over time and this property of the signal has been exploited quite effectively, to measure the distance of an object from a source. This model can be found in different sonar, radar, communications problems, see for example [1,6,3,5,9,8,12] and the references cited therein.

In practice, the parameters like amplitudes, frequency, frequency rate are unknown, and one tries to find efficient estimators of these unknown parameters, having some desired statistical properties. Recently, Kundu and Nandi [8]

* Corresponding author.
E-mail address: kundu@iitk.ac.in (D. Kundu).

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derived the asymptotic properties of the least squares estimators (LSEs), and it is observed that the LSEs are consistent and asymptotically normally distributed. It is further observed that the LSEs are efficient, with the convergence rates of the amplitude, frequency and frequency rate are $O_p(N^{-1/2})$, $O_p(N^{-3/2})$ and $O_p(N^{-5/2})$ respectively. Here $O_p(N^{-\delta})$ means $N^{\delta}O_p(N^{-\delta})$ is bounded in probability. Note that a sequence of random variables $\{X_n\}$ is said to be bounded in probability, if for any $\epsilon > 0$, there exists m, M, such that $P(m < X_n < M) \geq 1 - \epsilon$.

Unfortunately, Kundu and Nandi [8] did not discuss any estimation procedure of the LSEs. It is clear that they have to be obtained by some iterative procedure like Newton–Raphson method or its variant, see for example [13]. Although, the initial guesses and convergence of the iterative procedure seem to be important issues. It is well known, see [10], that even in the simple sinusoidal model, finding the LSEs is not an trivial issue. The problem is due to the fact that the least squares surface has several local minima, and therefore, if the initial estimators are not properly chosen, any iterative procedure will converge to a local minimum rather than the global minimum. Several methods have been suggested in the literature to find the efficient frequency estimators, see for example a very recent article by Kundu et al. [7] in this respect.

The main aim of this paper is to find estimators of the amplitudes, frequency and frequency rate efficiently, which have the same rates of convergence as the corresponding LSEs. It may be observed that the model (1) can be seen as a non-linear regression model, with A_0, B_0 as linear parameters, and α_0, β_0 as nonlinear parameters. If we can find efficient estimators of the nonlinear parameters α_0 and β_0, then efficient estimators of the linear parameters can be obtained by simple linear regression technique, see for example [11]. Because of this reason, in this paper we mainly concentrate on estimating the nonlinear parameters efficiently.

We propose an iterative procedure which has been applied to find efficient estimators of the frequency and frequency rate. It is observed that if we start the initial guesses of α_0 and β_0 with convergence rates $O_p(N^{-1})$ and $O_p(N^{-2})$respectively, then after four iterations the algorithm produces an estimate of α_0 with convergence rate $O_p(N^{-3/2})$, and an estimate of β_0 with convergence rate $O_p(N^{-5/2})$. Therefore, it is clear that the proposed algorithm produces estimates which have the same rates of convergence as the LSEs. Moreover, it is known that the algorithm stops after finite number of iterations.

We perform some simulation experiments, to see the effectiveness of the proposed method for different sample sizes and for different error variances. It is observed that the algorithm works very well. The mean squared errors (MSEs) of the proposed estimators are very close to the corresponding MSEs of the LSEs, and both are very close to the corresponding asymptotic variance of the LSEs. Therefore, the proposed method can be used very effectively instead of the LSEs. For illustrative purposes, we have analyzed one simulated data, and the performance is very satisfactory.

The rest of the paper is organized as follows. We provide the properties of the LSEs in Section 2. In Section 3, we present the proposed algorithm and provide the theoretical justification of the algorithm. The simulation results and the analysis of a simulated data have been presented in Section 4. Conclusions appear in Section 5. All the proofs are presented in the Appendix.

2. Existing results

In this section we present briefly the properties of the LSEs for ready reference. The LSEs of the unknown parameters of the model (1) can be obtained by minimizing $S(\mathbf{\Theta})$ with respect to $\mathbf{\Theta} = (A, B, \alpha, \beta)$, where

$$S(\mathbf{\Theta}) = \sum_{n=1}^{N} \left(y(n) - A\cos(\alpha n + \beta n^2) - B\sin(\alpha n + \beta n^2)\right)^2$$
$$= \left[\mathbf{Y} - \mathbf{W}(\alpha, \beta)\begin{bmatrix} A \\ B \end{bmatrix}\right]^T \left[\mathbf{Y} - \mathbf{W}(\alpha, \beta)\begin{bmatrix} A \\ B \end{bmatrix}\right].$$

Here $\mathbf{Y} = (y(1), \ldots, y(N))^T$, is the $N \times 1$ data vector and $\mathbf{W}(\boldsymbol{\theta})$ is the $N \times 2$ matrix of the following form;

$$\mathbf{W}(\boldsymbol{\theta}) = \begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ \cos(2\alpha + 4\beta) & \sin(2\alpha + 4\beta) \\ \vdots & \vdots \\ \cos(N\alpha + N^2\beta) & \sin(N\alpha + N^2\beta) \end{bmatrix}. \tag{3}$$

Note that, if α and β are known the LSEs of A_0 and B_0 can be obtained as $\widehat{A}(\boldsymbol{\theta})$ and $\widehat{B}(\boldsymbol{\theta})$ respectively, where $\boldsymbol{\theta} = (\alpha, \beta)$,

$$\left(\widehat{A}(\boldsymbol{\theta}), \widehat{B}(\boldsymbol{\theta})\right)^T = \left(\mathbf{W}^T(\boldsymbol{\theta})\mathbf{W}(\boldsymbol{\theta})\right)^{-1}\mathbf{W}^T(\boldsymbol{\theta})\mathbf{Y}. \tag{4}$$

Therefore, the LSEs of α_0 and β_0 can be obtained first by minimizing $Q(\alpha, \beta)$ with respect to α and β, where

$$Q(\alpha, \beta) = S(\widehat{A}(\boldsymbol{\theta}), \widehat{B}(\boldsymbol{\theta}), \alpha, \beta) = \mathbf{Y}^T\mathbf{W}(\boldsymbol{\theta})\left(\mathbf{W}^T(\boldsymbol{\theta})\mathbf{W}(\boldsymbol{\theta})\right)^{-1}\mathbf{W}^T(\boldsymbol{\theta})\mathbf{Y}. \tag{5}$$

Once the LSEs of α_0 and β_0, say $\widehat{\alpha}$ and $\widehat{\beta}$ are obtained the LSEs of A_0 and B_0 can be easily obtained as $\widehat{A}(\widehat{\alpha}, \widehat{\beta})$ and $\widehat{B}(\widehat{\alpha}, \widehat{\beta})$ respectively, see for example [11]. Kundu and Nandi [8] derived the properties of the LSEs and it is as follows. The LSEs