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# Least squares estimators for discretely observed stochastic processes driven by small Lévy noises

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ABSTRACT

We study the problem of parameter estimation for discretely observed stochastic processes driven by additive small Lévy noises. We do not impose any moment condition on the driving Lévy process. Under certain regularity conditions on the drift function, we obtain consistency and rate of convergence of the least squares estimator (LSE) of the drift parameter when a small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously. The asymptotic distribution of the LSE in our general setting is shown to be the convolution of a normal distribution and a distribution related to the jump part of the Lévy process. Moreover, we briefly remark that our methodology can be easily extended to the more general case of semi-martingale noises.



## 1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of $\sigma$-algebras $(\mathcal{F}_t, t \geq 0)$. Let $(L_t, t \geq 0)$ be an $\mathbb{R}^d$-valued Lévy process, which is given by

$$L_t = at + \sigma B_t + \int_0^t \int_{|z|\leq 1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|>1} zN(ds, dz), \tag{1.1}$$

where $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, $\sigma = (\sigma_{ij})_{d\times r}$ is a $d \times r$ real-valued matrix, $B_t = (B_t^1, \ldots, B_t^r)$ is an $r$-dimensional standard Brownian motion, $N(ds, dz)$ is an independent Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$ with characteristic measure $dt\nu(dz)$, and $\tilde{N}(ds, dz) = N(ds, dz) - \nu(dz)ds$ is a martingale measure. Here we assume that $\nu(dz)$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d\setminus\{0\}}(|z|^2 \wedge 1)\nu(dz) < \infty$ with $|z| = \sqrt{\sum_{i=1}^d z_i^2}$. The stochastic process $X = (X_t, t \geq 0)$, starting from $x_0 \in \mathbb{R}^d$, is defined as the unique strong solution to the following stochastic differential equation (SDE)

$$dX_t = b(X_t, \theta)dt + \varepsilon dL_t, \quad t \in [0, 1]; \qquad X_0 = x_0, \tag{1.2}$$

where $\theta \in \Theta = \bar{\Theta}_0$ (the closure of $\Theta_0$) with $\Theta_0$ being an open bounded convex subset of $\mathbb{R}^p$, and $b = (b_1, \ldots, b_d) : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$ is a known function. Without loss of generality, we assume that $\varepsilon \in (0, 1]$. The regularity conditions on $b$ will

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be provided in Section 2. Assume that this process is observed at regularly spaced time points $\{t_k = k/n,\ k = 1, 2, \ldots, n\}$. The only unknown quantity in SDE (1.2) is the parameter $\theta$. Let $\theta_0 \in \Theta_0$ be the true value of the parameter $\theta$. The purpose of this paper is to study the least squares estimator for the true value $\theta_0$ based on the sampling data $(X_{t_k})_{k=1}^n$ with small dispersion $\varepsilon$ and large sample size $n$.

In the case of diffusion processes driven by Brownian motion, a popular method is the maximum likelihood estimator (MLE) based on the Girsanov density when the processes can be observed continuously (see Prakasa Rao [31], Liptser and Shiryaev [19], Kutoyants [16], and Bishwal [2]). When a diffusion process is observed only at discrete times, in most cases the transition density and hence the likelihood function of the observations is not explicitly computable. In order to overcome this difficulty, some approximate likelihood methods have been proposed by Lo [20], Pedersen [27,28], Poulsen [29], and Aït-Sahalia [1]. For a comprehensive review on MLE and other related methods, we refer to Sørensen [37]. The least squares estimator (LSE) is asymptotically equivalent to the MLE. For the LSE, the convergence in probability was proved in Dorogovcev [5] and Le Breton [18], the strong consistency was studied in Kasonga [12], and the asymptotic distribution was studied in Prakasa Rao [30]. For a more recent comprehensive discussion, we refer to Prakasa Rao [31], Kutoyants [16], Bishwal [2] and the references therein.

The parametric estimation problems for diffusion processes with jumps based on discrete observations have been studied by Shimizu and Yoshida [35] and Shimizu [33] via the quasi-maximum likelihood. They established consistency and asymptotic normality for the proposed estimators. Moreover, Ogihara and Yoshida [26] showed some stronger results than the ones by Shimizu and Yoshida [35], and also investigated an adaptive Bayes-type estimator with its asymptotic properties. The driving jump processes considered in Shimizu and Yoshida [35], Shimizu [33] and Ogihara and Yoshida [26] include a large class of Lévy processes such as compound Poisson processes, gamma, inverse Gaussian, variance gamma, normal inverse Gaussian or some generalized tempered stable processes. Masuda [24] dealt with the consistency and asymptotic normality of the TFE (trajectory-fitting estimator) and LSE when the driving process is a zero-mean adapted process (including Lévy process) with finite moments. The parametric estimation for Lévy-driven Ornstein–Uhlenbeck processes was also studied by Brockwell et al. [3], Spiliopoulos [39], and Valdivieso et al. [46]. However, the aforementioned papers were unable to cover an important class of driving Lévy processes, namely $\alpha$-stable Lévy motions with $\alpha \in (0, 2)$. Recently, Hu and Long [9,10] have started the study on parameter estimation for Ornstein–Uhlenbeck processes driven by $\alpha$-stable Lévy motions. They obtained some new asymptotic results on the proposed TFE and LSE under continuous or discrete observations, which are different from the classical cases where asymptotic distributions are normal. Fasen [6] extended the results of Hu and Long [10] to multivariate Ornstein–Uhlenbeck processes driven by $\alpha$-stable Lévy motions. Masuda [25] proposed a self-weighted least absolute deviation estimator for discretely observed ergodic Ornstein–Uhlenbeck processes driven by symmetric Lévy processes.

The asymptotic theory of parametric estimation for diffusion processes with small white noise based on continuous-time observations has been well developed (see, e.g., Kutoyants [14,15], Yoshida [48,50], Uchida and Yoshida [44]). There have been many applications of small noise asymptotics to mathematical finance, see for example Yoshida [49], Takahashi [40], Kunitomo and Takahashi [13], Takahashi and Yoshida [41], Uchida and Yoshida [45]. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for diffusion processes with small noise based on discrete observations. Substantial progress has been made in this direction. Genon-Catalot [7] and Laredo [17] studied the efficient estimation of drift parameters of small diffusions from discrete observations when $\varepsilon \to 0$ and $n \to \infty$. Sørensen [36] used martingale estimating functions to establish consistency and asymptotic normality of the estimators of drift and diffusion coefficient parameters when $\varepsilon \to 0$ and $n$ is fixed. Sørensen and Uchida [38] and Gloter and Sørensen [8] used a contrast function to study the efficient estimation for unknown parameters in both drift and diffusion coefficient functions. Uchida [42,43] used the martingale estimating function approach to study estimation of drift parameters for small diffusions under weaker conditions. Thus, in the cases of small diffusions, the asymptotic distributions of the estimators are normal under suitable conditions on $\varepsilon$ and $n$.

Long [21] studied the parameter estimation problem for discretely observed one-dimensional Ornstein–Uhlenbeck processes with small Lévy noises. In that paper, the drift function is linear in both $x$ and $\theta$ ($b(x, \theta) = -\theta x$), the driving Lévy process is $L_t = aB_t + bZ_t$, where $a$ and $b$ are known constants, $(B_t, t \geq 0)$ is the standard Brownian motion and $Z_t$ is a $\alpha$-stable Lévy motion independent of $(B_t, t \geq 0)$. The consistency and rate of convergence of the least squares estimator are established. The asymptotic distribution of the LSE is shown to be the convolution of a normal distribution and a stable distribution. In a similar framework, Long [22] discussed the statistical estimation of the drift parameter for a class of SDEs with special drift function $b(x, \theta) = \theta b(x)$. Ma [23] extended the results of Long [21] to the case when the driving noise is a general Lévy process. However, all the drift functions discussed in Long [21,22] and Ma [23] are linear in $\theta$, which restricts the applicability of their models and results. In this paper, we allow the drift function $b(x, \theta)$ to be nonlinear in both $x$ and $\theta$, and the driving noise to be a general Lévy process. We are interested in estimating the drift parameter in SDE (1.2) based on discrete observations $\{X_{t_i}\}_{i=1}^n$ when $\varepsilon \to 0$ and $n \to \infty$. We shall use the least squares method to obtain an asymptotically consistent estimator.

Consider the following *contrast function*

$$\Psi_{n,\varepsilon}(\theta) = \sum_{k=1}^{n} \frac{|X_{t_k} - X_{t_{k-1}} - b(X_{t_{k-1}}, \theta) \cdot \Delta t_{k-1}|^2}{\varepsilon^2 \Delta t_{k-1}},$$