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Estimation of means of multivariate normal populations with order restriction[☆]

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ABSTRACT

Multivariate isotonic regression theory plays a key role in the field of statistical inference under order restriction for vector valued parameters. Two cases of estimating multivariate normal means under order restricted set are considered. One case is that covariance matrices are known, the other one is that covariance matrices are unknown but are restricted by partial order. This paper shows that when covariance matrices are known, the estimator given by this paper always dominates unrestricted maximum likelihood estimator uniformly, and when covariance matrices are unknown, the plug-in estimator dominates unrestricted maximum likelihood estimator under the order restricted set of covariance matrices. The isotonic regression estimators in this paper are the generalizations of plug-in estimators in unitary case.

1. Introduction

Let X_{ij} be the jth $p \times 1$ observation vector of the ith population and mutually independently distributed as $N(\mu_i, \Sigma_i)$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, where μ_i and Σ_i are unknown. Let $\preceq$ be a partial order on the index set $\{1, \ldots, k\}$. Let μ_{il} be the lth element of vector μ_i. The vector set $\mu_1, \ldots, \mu_k$ is said to be isotonic with respect to $\preceq$ in the Löwner sense means that if $j \preceq i$, then $\mu_{il} \geq \mu_{jl}$, $l = 1, \ldots, p$, which is written as $\mu_i \succeq \mu_j$. Define

$$\Omega = \{\mu : \mu_1 \preceq \mu_2 \preceq \cdots \preceq \mu_k\}. \tag{1.1}$$

For convenience, we still use $\mu_j \leq \mu_i$ to denote $\mu_j \preceq \mu_i$ in the following state.

If $p = 1$, the above problem reduces to univariate case and Σ_i reduces to σ_i^2. The estimation problem of two normal means was discussed by [1]. Many authors studied such problem when $k > 2$. When all variances are known and $\mu_1 = \mu_2 = \cdots = \mu_k = \mu$, a natural unbiased estimator of μ is

$$\hat{\mu} = \frac{\sum_{j=1}^{k} \left(\frac{n_j}{\sigma_j^2}\right) \bar{X}_j}{\sum_{j=1}^{k} \left(\frac{n_j}{\sigma_j^2}\right)}.$$

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In practice, variances are often unknown, so a feasible estimator is Graybill-Deal estimator

$$\hat{\mu}_{GD} = \frac{\sum_{j=1}^{k}\left(\frac{n_j}{\hat{\sigma}_j^2}\right)\bar{X}_j}{\sum_{j=1}^{k}\left(\frac{n_j}{\hat{\sigma}_j^2}\right)}. \tag{1.2}$$

Misra and van der Meulen [2] extended the result given by [3] and [4] and proposed an isotonic estimator of μ

$$\hat{\mu}_T = \frac{\sum_{i=1}^{k} n_{k-i+1}\hat{\tau}_i\bar{X}_{k-i+1}}{\sum_{i=1}^{k} n_{k-i+1}\hat{\tau}_i}, \tag{1.3}$$

which is better than the usual Graybill-Deal estimator in terms of stochastic dominance and the Pitman measure of closeness over a simple order restriction in terms of

$$0 < \sigma_1^2 \le \sigma_2^2 \le \cdots \le \sigma_k^2 < \infty, \tag{1.4}$$

where $\hat{\tau}_i = \min_{t\ge i}\max_{s\le i}((\sum_{r=s}^{t} n_{k-r+1})^{-1}\sum_{r=s}^{t} n_{k-r+1}P_r)$, $P_i = \frac{1}{S_{k-i+1}}$, $S_i = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2$.

Under (1.1), Robertson et al. [5] gave the restricted maximum likelihood estimator of μ_i

$$\min_{t\ge i}\max_{s\le i}\frac{\sum_{j=s}^{t}\left(\frac{n_j}{\sigma_j^2}\right)\bar{X}_j}{\sum_{j=s}^{t}\left(\frac{n_j}{\sigma_j^2}\right)}. \tag{1.5}$$

Lee [6] and Kelly [7] showed that the estimator (1.5) uniformly improves upon $\bar{X}_i$. Shi and Jiang [8] discussed some properties of the maximum likelihood estimates of means and unknown variances under the restriction and proposed an algorithm for obtaining estimates.

If $k = 2$, [1] gave the plug-in estimators of μ_i which uniformly improves upon $\bar{X}_i$ under set (1.1) when σ_i are unknown.

The above discussions are simple unitary estimation problems. In multivariate case i.e. $p \ge 2$, Chiou and Cohen [9]showed that

$$\hat{\mu}_{GDd} = \left(\sum_{i=1}^{k} n_i S_i^{-1}\right)^{-1}\sum_{i=1}^{k} n_i S_i^{-1}\bar{X}_j, \tag{1.6}$$

dominates neither $\bar{X}_1$ nor $\bar{X}_2$, when $\mu_1 = \mu_2 = \cdots = \mu_k = \mu$ and $k = 2$. Loh [10] estimated the mean vector from a symmetric loss function point of view as alternatives to $\hat{\mu}_{GDd}$.

Jordan and Krishnamoorthy [11] considered a weighted Graybill-Deal estimator

$$\hat{\mu}_{GDW} = \left(\sum_{i=1}^{k} c_i n_i S_i^{-1}\right)^{-1}\sum_{i=1}^{k} c_i n_i S_i^{-1}\bar{X}_j, \tag{1.7}$$

where $c_i = \frac{[Var(T_i^2)]^{-1}}{\sum_{i=1}^{k}[Var(T_i^2)]^{-1}}$, $T_i^2 = \sum_{i=1}^{k} n_i(\bar{X}_i-\mu)'S_i^{-1}(\bar{X}_i-\mu)$ and $Var(T_i^2) = \frac{2p(n_i-1)^2(n_i-2)}{(n_i-p-2)^2(n_i-p-4)}$.

While the isotonicity of unitary normal populations has been the primary concern in the past studies, very little attention has been paid to the isotonic estimation problem of multivariate normal means. It is obvious that Graybill-Deal estimator play an important role in establishing the isotonic estimator of normal means whether in univariate case or multivariate case. The isotonic estimation of multivariate normal means is also based on Graybill-Deal idea in this paper when mean vectors are restricted by simple order.

In order to obtain the dominance results of estimators, we consider the squared error loss of the estimators of μ_i,

$$L(\mu_i, \hat{\mu}_i, D) = (\hat{\mu}_i - \mu_i)'D(\hat{\mu}_i - \mu_i), \tag{1.8}$$

where D is a positive definite matrix.

Then the risk is given by

$$R(\mu_i, \hat{\mu}_i, D) = E[L(\mu_i, \hat{\mu}_i, D)]. \tag{1.9}$$

The main object of this paper is to generalize dominance results given by [1] to multivariate case and consider the problem of estimating the mean vectors of two normal populations under order restriction. When Σ_i is known, we give a restricted maximum likelihood estimator which is similar to the equivalent in univariate case and show that it improves upon the unrestricted maximum likelihood estimator uniformly. When Σ_i is unknown, we obtain a feasible plug-in estimator dominating the unrestricted maximum likelihood estimator under order restricted set of covariance matrices respectively. These results are derived in Section 2. In Section 3, the proof processes of theoretical results of Section 2 are given.