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Characteristic functions of scale mixtures of multivariate skew-normal distributions

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ABSTRACT

We obtain the characteristic function of scale mixtures of skew-normal distributions both in the univariate and multivariate cases. The derivation uses the simple stochastic relationship between skew-normal distributions and scale mixtures of skew-normal distributions. In particular, we describe the characteristic function of skew-normal, skew-*t*, and other related distributions.

1. Introduction

Characteristic functions play an important role in statistics. It is well-known that every bounded and measurable function is integrable with respect to any distribution over the real line. This assures the existence of the characteristic function for any distribution function. Specifically, the characteristic function of a random variable X is the function $\Psi_X : \mathbb{R} \to \mathbb{C}$ defined as

$$\Psi_X(t) = \mathrm{E}\{\exp(\mathrm{i}tX)\}, \quad t \in \mathbb{R}, \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$. An important result is that there is a one-to-one correspondence between the distribution function of a random variable and its characteristic function. This leads to derivations of distributions of many statistics, for example of sums of independent random variables. Moreover, by the Cramér–Wold theorem, a multivariate distribution is uniquely determined by the distributions of all linear combinations of the component variables. By the continuity theorem, the one-to-one correspondence between distribution functions and characteristic functions is continuous. Furthermore, characteristic functions have many pleasant properties including that every characteristic function is uniformly continuous on the whole real line. More properties of characteristic functions can be found in [21,22,12].

In recent years, there has been a growing interest in the construction of parametric classes of non-normal distributions. For instance, the univariate skew-normal distribution has been developed by Azzalini [6] and then further extended to

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the multivariate setting by Azzalini and Dalla Valle [10] and Azzalini and Capitanio [8]. Scale mixtures of skew-normal distributions have appeared in [13] and include (skew-) normal distributions as special cases. They have been further extended to skew-elliptical distributions by many authors, see for example [15] and references therein. All these skewed distributions can be cast in the framework of selection distributions that arise under various selection mechanisms, see [3].

The moment generating function has been studied by Arellano-Valle and Genton [4] for fundamental skew-symmetric distributions, by Arellano-Valle and Azzalini [2] for unified skew-normal distributions, by Arellano-Valle et al. [3] for scale mixtures of skew-normal distributions, and by Arellano-Valle and Genton [5] for quadratic forms in random vectors with a selection distribution. Although characteristic functions have nice properties and determine the distribution function uniquely, there have been surprisingly few research articles on the characteristic functions of the aforementioned skewed distributions. Pewsey [24] developed the characteristic function of the univariate skew-normal distribution. Kozubowski and Podgórski [19] discussed the origins and inter-relations of various skew-Laplace distributions. In doing so, they presented the characteristic functions of some skew-Laplace distributions. Therefore, there is a need to develop characteristic functions of these skewed distributions, both in the univariate and multivariate cases. We concentrate on the characteristic functions of scale mixtures of multivariate skew-normal distributions which include some well-known distributions, for example such as the skew-normal, skew-t, and skew-slash [27] distributions among others.

This paper is organized as follows. In Section 2, we review scale mixtures of multivariate skew-normal distributions and their properties. In Section 3, we first derive the characteristic function of the univariate skew-normal distribution and prove that the derived formula is the same as the one given by Pewsey [24]. We give also another form of that characteristic function and various methods of proof. The characteristic function of the skew-t distribution is obtained using scale mixtures of skew-normal distributions and the Laplace–Stieltjes transform. Using a lemma by Azzalini [7], we derive the characteristic function of scale mixtures of skew-normal distributions in a series representation when the mixing moments exist. Generally, we derive the characteristic function of scale mixtures of skew-normal distributions as mixtures of characteristic functions of skew-normal distributions. In Section 4, the characteristic functions of multivariate skew-normal and skew-t distributions are obtained. Finally, the characteristic function of scale mixtures of multivariate skew-normal distributions is obtained as a mixture of the characteristic functions of multivariate skew-normal distributions.

2. Scale mixtures of skew-normal distributions

2.1. Univariate case

A random variable Z has a skew-normal distribution developed by Azzalini [6] if its probability density function (pdf) is

$$f_Z(z) = 2\phi(z)\Phi(\alpha z), \quad z \in \mathbb{R}, \tag{2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cumulative distribution function (cdf) of the standard normal $N(0, 1)$ distribution, respectively. The parameter $\alpha \in \mathbb{R}$ controls the skewness (shape) of the distribution. When Z has the pdf (2), we write $Z \sim \mathrm{SN}(\alpha)$. If $\alpha = 0$, then (2) reduces to the $N(0, 1)$ pdf. The location-scale extension of Z is

$$X = \xi + \omega Z, \tag{3}$$

with pdf

$$f_X(x) = 2\phi(x; \xi, \omega^2)\Phi\left(\alpha\frac{x-\xi}{\omega}\right), \quad x \in \mathbb{R}, \tag{4}$$

where $\phi(\cdot; \xi, \omega^2)$ denotes the normal pdf with location $\xi \in \mathbb{R}$ and scale $\omega > 0$. In this case, we denote the skew-normal random variable $X \sim \mathrm{SN}(\xi, \omega^2, \alpha)$, similarly to Azzalini and Capitanio [8].

Scale mixtures of skew-normal distributions appeared in [13] and are defined by the following stochastic representation:

$$Y = \xi + W(\eta)^{1/2}X, \tag{5}$$

where X has the skew-normal distribution $\mathrm{SN}(0, \omega^2, \alpha)$ and η is a mixing variable with cdf $H(\eta)$ and a weight function $W(\eta)$, independent of X. If X has a normal distribution, i.e. $X \sim N(0, \omega^2)$, then (5) is the stochastic representation of scale mixtures of normal distributions.

Consequently, the pdf of Y is

$$f_Y(y) = 2\int_0^\infty \phi(y; \xi, W(\eta)\omega^2)\Phi\left(\alpha\frac{y-\xi}{W(\eta)^{1/2}\omega}\right)\mathrm{d}H(\eta) \tag{6}$$

where $H(\eta)$ is the cdf of η. One particular case of this distribution is the skew-normal distribution, for which H is degenerate, with $W(\eta) = 1$. In this case the pdf (6) reduces to (4). We summarize special cases of scale mixtures of skew-normal distributions in Table 1 along with their mixing pdf $h(\eta)$. These mixing distributions also apply to multivariate