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Asymptotic distribution for periodograms of infinite dimensional discrete time periodically correlated processes

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ABSTRACT

In this article we shall consider a class of strongly T-periodically correlated processes with values in a separable complex Hilbert space $\mathbf{H}$. The periodograms of these processes and their statistical properties were previously studied by the authors. In this paper we derive the asymptotic distribution of the periodogram, that appears to be a certain Wishart distribution on $\mathbf{H}^T$.

1. Introduction

Let $\mathbf{H}$ be a separable complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle_{\mathbf{H}}$ and $\mathcal{B}$ denote the Borel field in $\mathbf{H}$. Also let $(\Omega, \mathcal{F}, P)$ stand for an underlying probability space. A random element in $\mathbf{H}$ is an $\mathcal{F}/\mathcal{B}$ measurable mapping from Ω into $\mathbf{H}$. A sequence of random elements in $\mathbf{H}$, namely $\boldsymbol{\xi} = \{\xi_n,\ n \in \mathbb{Z}\}$, $\mathbb{Z}$ is the set of integers, is called an $\mathbf{H}$-valued discrete time stochastic process. We use the abbreviation $\mathbf{H}$-VSP, and drop discrete time whenever there is no ambiguity. Let us denote by $L^2(\Omega, \mathcal{F}, P)$ the Hilbert space of all complex random variables on the probability space $(\Omega, \mathcal{F}, P)$ with zero mean and finite variance equipped with the inner product $E(X\overline{Y}) = \int_\Omega X(\omega)\overline{Y(\omega)}\mathrm{d}P(\omega)$. A random element ξ in $\mathbf{H}$ is said to be weakly second order if $E|\langle\xi, x\rangle_{\mathbf{H}}|^2 < \infty$ for every $x \in \mathbf{H}$, and strongly second order if $E\|\xi\|^2_{\mathbf{H}} < \infty$. Evidently the weak second order property is weaker than the strong second order. Similarly, an $\mathbf{H}$-VSP $\boldsymbol{\xi}$ is said to be weakly second order ($\mathbf{H}$-V[WSO]SP in abbreviation) if every ξ_n is weakly second order, and strongly second order ($\mathbf{H}$-V[SSO]SP in abbreviation) if every ξ_n is strongly second order.

We let $\boldsymbol{\xi}$ be an $\mathbf{H}$-V[WSO]SP. Interestingly, the linear mapping $x \to \langle\xi, x\rangle_{\mathbf{H}} = \xi_x$ from $\mathbf{H}$ into $L^2(\Omega, \mathcal{F}.P)$ has a closed graph. Consequently, by the closed graph theorem, it is bounded and $E\langle\xi_n, x\rangle_{\mathbf{H}}\overline{\langle\xi_m, y\rangle_{\mathbf{H}}}$ is a bilinear functional on $\mathbf{H} \times \mathbf{H}$, which is bounded in the sense that $|E\langle\xi_n, x\rangle_{\mathbf{H}}\overline{\langle\xi_m, y\rangle_{\mathbf{H}}}| < c\|x\|\,\|y\|$, for a constant c. Thus, by Theorem 12.8 in [9] there exists a bounded linear operator $\mathbf{C}_\xi^{n,m}$ on $\mathbf{H}$ such that

$$E\xi_{n,x}\overline{\xi_{m,y}} = \langle\mathbf{C}_\xi^{n,m}x, y\rangle_{\mathbf{H}}. \tag{1}$$

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Then $\mathbf{C} = \{\mathbf{C}^{n,m},\ n, m, \in \mathbb{Z}\}$ is called the covariance of the **H**-V[WSO]SP $\boldsymbol{\xi}$. An **H**-V[WSO]SP $\boldsymbol{\xi}$ is said to be periodically correlated (PC) if there exists a positive integer T such that

$$\mathbf{C}^{n,m} = \mathbf{C}^{n+T,m+T}. \tag{2}$$

The smallest T in (2) is called the period of the process $\boldsymbol{\xi}$.

In this article we provide the asymptotic distributions for the finite Fourier transform (FFT) and periodogram of a PC **H**-V[SSO]SP $\boldsymbol{\xi}$; these notions are defined in the section of preliminaries. Indeed in the theory of inference and spectral estimation for stationary stochastic processes, the FFT and periodogram are essential and basic tools, and are widely used. The asymptotic distributions of the FFT and the periodogram for univariate discrete time stationary processes are given in [1], for multivariate stationary processes in [3]. For non-stationary processes, the spectral distribution in general is not properly defined. But there are exceptions. For harmonizable processes, the spectral distribution is indeed a bi-measure, and in certain cases it becomes a positive measure defined on certain product spaces; in this case, the FFT can be used to detect active frequencies. Discrete time periodically correlated processes are harmonizable and assume such spectral distributions. For the univariate PC processes, the FFT and corresponding periodogram are formulated, studied, and their asymptotic distributions are derived by Soltani and Azimmohseni [11].

Recent developments on operatorial statistics and analysis of functional data [2] have stimulated research on estimation and statistical inference for infinite dimensional processes. Periodograms are for detecting active frequencies; and their limiting distributions ease statistical inference on unknown spectral densities. Spectral analysis in general and periodograms in particular are valuable tools for model building. Functional data is rapidly advancing due to the correct understanding that real phenomena are hardly caused by a finite number of parameters. This research is expected to enrich the spectral theory and periodogram analysis on functional data.

The FFT and periodogram for Hilbert space-valued discrete time processes are considered by the authors in [14], where it is proved that the periodogram is an asymptotically unbiased estimator for the spectral density. In this paper we proceed further and derive the asymptotic distributions of the FFT and periodogram of these processes.

Our methodology relies on using the central limit theorem (CLT) for sequences of independent random elements in complex Hilbert spaces. This theorem exists in the literature for real Hilbert spaces [5]. But, to the best of our knowledge, the theorem is not produced for complex Hilbert spaces. In Section 3, using the [15] isometry between real and complex Hilbert spaces, we rephrase the cited CLT for the complex Hilbert spaces. Then we derive our desired asymptotic distributions, under the assumption of the continuity of the Cholesky factor of the spectral density, by applying the CLT to the underlying innovations, that are assumed to be independent. Section 4 is devoted to the main results of the article on the periodograms' asymptotic distributions. In Section 5, we provide certain PC **H**-V[SSO] autoregressive processes of finite and infinite orders and derive their FFT and the asymptotic distribution of the periodograms.

2. Preliminaries

Let **H** denote a separable complex Hilbert space, and $\mathsf{L}(\mathbf{H})$ stand for the Banach space of all bounded linear operators on **H**.

There are two spectral representations for a PC **H**-V[WSO]SP $\boldsymbol{\xi} = \{\xi_n,\ n \in \mathbb{Z}\}$; the first is similar to that of Gladyshev and the second to that of Soltani.

These representations are given in [13], namely,

$$\xi_n = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ns}\mathcal{Z}(\mathrm{d}s), \tag{3}$$

where the spectral random measure $\mathcal{Z}$ does not necessarily have orthogonal increments, however it does possess the property that $E\mathcal{Z}(\mathrm{d}s)x\overline{\mathcal{Z}(\mathrm{d}\lambda)y} = \langle x, F(\mathrm{d}s, \mathrm{d}\lambda)y\rangle_{\mathbf{H}}$ defines a so-called spectral distribution $F(\cdot,\cdot)$ on $[0, 2\pi) \times [0, 2\pi)$, which is supported by the parallel lines $d_k = \{(s, \lambda) \in [0, 2\pi)^2,\ s - \lambda = \frac{2\pi k}{T}\},\ k = 1 - T, \dots, T - 1$. The spectral distribution is indeed specified by T, $\mathsf{L}(\mathbf{H})$-valued distributions $\{F_0, F_1, \dots, F_{T-1}\}$ on $[0, 2\pi)$ such that the matrix $\mathcal{F}$ defined by $\mathcal{F}(\mathrm{d}s) = (F_{p-l}(\mathrm{d}s + \frac{2\pi p}{T}))_{p,l=0,\dots,T-1},\ s \in [0, \frac{2\pi}{T})$, is positive definite; see [6,13]. We also assume that the spectral density $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}(\mathrm{d}s)$ exists:

$$\mathbf{f}(s) = \left(f_{p-l}\left(\mathrm{d}s + \frac{2\pi p}{T}\right)\right)_{l,p=0,\dots,T-1}, \quad s \in \left[0, \frac{2\pi}{T}\right). \tag{4}$$

The alternative spectral representation that is heavily used in this article provides a time dependent spectral representation for the PC **H**-V[WSO]SP $\boldsymbol{\xi}$, namely

$$\xi_{n,x} = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ns}V_n(s)[x]\Phi(\mathrm{d}s), \quad x \in \mathbf{H}, \tag{5}$$

in the sense that $E\xi_{n,x}\overline{\xi_{m,y}} = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-m)s}\langle V_n(s)x, V_m(s)y\rangle_{\mathbf{H}}\mathrm{d}s$, where Φ is a random spectral with orthogonal increments, and $V_n(s) = \sum_{k=0}^{T-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi kn}{T}} a_k(s + \frac{2\pi k}{T})$ is a sequence of T-periodic $\mathsf{L}(\mathbf{H})$-valued functions for $s \in [0, 2\pi)$ and $n \in \mathbb{Z}$. Furthermore,