Journal of Multivariate Analysis 101 (2010) 1168–1178

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Wishart–Laplace distributions associated with matrix quadratic forms[☆]

Joe Masaro [a], Chi Song Wong [b,∗]

[a] 961 Marentette Ave., Windsor, ON, Canada N9A 2A2
[b] University of Windsor, Windsor, Ontario, Canada N9B 3P4

ARTICLE INFO

Article history:
Received 29 January 2009
Available online 29 December 2009

AMS 2000 subject classifications:
primary 62H05
secondary 62H10

Keywords:
Matrix quadratic form
Laplacian distribution
Wishart distribution
Jordan algebras
Multivariate normal matrix
Cochran theorem

ABSTRACT

For a normal random matrix Y with mean zero, necessary and sufficient conditions are obtained for $Y'W_kY$ to be Wishart–Laplace distributed and $\{Y'W_kY\}$ to be independent, where each W_k is assumed to be symmetric rather than nonnegative definite.

1. Introduction

Let Y be an $n \times p$ normal random matrix with mean zero and covariance Σ_Y and $\{W_k\}$ be a family of $n \times n$ symmetric matrices. In this paper, we shall obtain a very general Cochran theorem, i.e., obtain necessary and sufficient conditions for $\{Y'W_kY\}$to be independent and each $Y'W_kY$ to be Wishart–Laplace $DW_p(m_1, m_2, \Sigma)$, i.e., $Y'W_kY$ is distributed as the difference $Q_1 - Q_2$, where Q_1 and Q_2 are independent (central) Wishart $W_p(m_1, \Sigma)$ and $W_p(m_2, \Sigma)$ distributed random matrices.

For a brief history, [1] solves the above problem by assuming that $m_2 = 0, W_k$ is nonnegative definite and Σ is nonsingular. Wong and Wang [2] remove the nonsingularity condition on Σ so that the result can be applied to multivariate components of variance models; see [3–6]. Mathai [7] extended a chi-squared version to one that includes the family of Laplace distributions; its multivariate version is our version for a family of Wishart–Laplace distributions; see Wong and Wang [8]. For the case W_k is symmetric, [9] obtained necessary and sufficient conditions for $\{Y'W_kY\}$ to be independent; [10] obtained necessary and sufficient conditions for $Y'W_kY$ to be $W_p(m_k, \Sigma)$. In Theorem 2.1, we generalize these results to that for Wishart–Laplace distributions. The result is expected to be more cumbersome; this is partially caused by our not using appropriate multiplication. Since the sample covariance Σ_Y and the population covariance Σ are both symmetric, we should develop the theory needed within the set, $\mathcal{S}_N$, of $N \times N$ symmetric matrices, where N may be p, np, or some other positive integer. Let $A, B \in \mathcal{S}_N$. The usual matrix product AB may not be symmetric. So we introduce a Jordan product:

$$A * B = \frac{1}{2}(AB + BA),$$

☆ Research supported by NSERC Grant of Canada, PIN7914.
∗ Corresponding author.
E-mail addresses: jmasaro1@cogeco.ca (J. Masaro), whynanhai@yahoo.com (C.S. Wong).

0047-259X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2009.12.013

or more generally,

$$A *_C B = \frac{1}{2}(ACB + BCA),$$

where $C \in \mathcal{S}_N$. Then $(\mathcal{S}_N, *_C)$ is an example of a Jordan algebra; see, e.g., [11]. Now consider the distribution of $Q_Y = Y'WY$ through its moment generating function M_{Q_Y}, where W is symmetric. From [12],

$$M_{Q_Y}(t) = Det\ [I_n \otimes I_p - 2\Sigma_Y^{1/2}(W \otimes t)\Sigma_Y^{1/2}]^{-1/2}, \quad t \in \mathcal{N}_\circ,$$

where $\mathcal{N}_\circ = \{t \in \mathcal{S}_p : I_n \otimes I_p - 2\Sigma_Y^{1/2}(W \otimes t)\Sigma_Y^{1/2}$ is positive definite$\}$, i.e., M_{Q_Y} is determined by the linear map $h : \mathcal{S}_p \to \mathcal{S}_{np}$ with

$$h(t) = \Sigma_Y^{1/2}(W \otimes t)\Sigma_Y^{1/2}, \quad t \in \mathcal{S}_p,$$

or the more flexible map $\rho : \mathcal{S}_p \to \mathcal{S}_q$ with

$$\rho(t) = M(W \otimes t)M', \quad t \in \mathcal{S}_p,$$

where $\Sigma_Y = M'M$, $M \in M_{q\times np}$, $q \leq np$. It can be proved (see [12]) that $Y'WY$ is Wishart with nonsingular scale parameter Σ if and only if ρ is homomorphic from $(\mathcal{S}_p, *_\Sigma)$ into $(\mathcal{S}_q, *)$, i.e., it preserves the Jordan products:

$$\rho(A *_\Sigma B) = \rho(A) * \rho(B), \quad A, B \in \mathcal{S}_p.$$

The mappings ρ and h are referred to as representations in (Jordan) algebra.

Our main result, Theorem 2.1, extends this result to a result for the Wishart–Laplace distribution. Also both Theorems 2.1 and 2.3 may be extended to the case where $Y \sim N_{n\times p}(\mu, \Sigma_Y)$ and $\mu \neq 0$. This extension is simply a matter of adding extra conditions in [9,8].

We remark that the use of representation theory can hardly be over emphasized. Through abstraction, Jordan algebra and its representations can be much more general (see [13,11,14–18,12]). For the present paper, Jordan algebras of the type $\mathcal{S}_N$ with Jordan products $*$ or $*_C$ are enough. Our results may be extended to the case where Y is complex normal (or even the quaternion normal); this requires the general theory of Jordan algebras and is beyond the scope of the present paper (see [19]).

Below is a summary of the notation used in what follows.

$\mathcal{M}_{a\times b}$: the family of $a \times b$ matrices over $\Re$.
$\mathcal{S}_a$: the family of symmetric $a \times a$ matrices over $\Re$.
$\mathcal{N}_a$: the family of nonnegative definite $a \times a$ matrices over $\Re$.
$A^{1/2}$: for $A \in \mathcal{N}_a$, $A^{1/2}$ will always be the unique nonnegative definite square root of A.
A^+: the Moore–Penrose inverse of a matrix $A \in \mathcal{M}_{a\times b}$.
A°: for $A \in \mathcal{M}_{a\times b}$, $A^\circ = AA^+$ is the orthogonal projection of $\Re^a$ onto the image of A.
$\langle , \rangle$: the trace inner product on $\mathcal{M}_{a\times b}$, $\langle A, B\rangle = tr(AB')$.
$\| \|$: the trace norm on $\mathcal{M}_{a\times b}$, $\|A\|^2 = \langle A, A\rangle$.
$A \otimes B$: for $A \in \mathcal{M}_{a\times b}$, $B \in \mathcal{M}_{c\times d}$, $A \otimes B = [a_{ij}B] \in \mathcal{M}_{ac\times bd}$.
$(\mathcal{S}_a, *)$: the family $\mathcal{S}_a$ with the Jordan product $A * B = \frac{1}{2}(AB + BA)$.
$(\mathcal{S}_a, *_C)$: the family $\mathcal{S}_a$ with the Jordan product $A *_C B = \frac{1}{2}(ACB + BCA)$.
$X^{m(*_C)}$: for X in $\mathcal{S}_a$, $X^{m(*_C)}$ represents X to the power m with respect to the Jordan product $*_C$: $X^{m(*_C)} = XCXCXCX \ \ldots\ XCX$ with X appearing m times in the product.
K_{np}: the commutation matrix K_{np} has the following basic properties: $K'_{np} = K_{pn}$, $K_{np}K'_{np} = I_{np}$, $K_{np}(A \otimes B)K_{qr} = B \otimes A$, $A \in \mathcal{M}_{p\times q}$, $B \in M_{n\times r}$, and for a random $n \times p$ matrix Y, $\Sigma_{Y'} = K'_{np}\Sigma_Y K_{np}$; see, for example, [20].
$Y \sim N_{n\times p}(\mu, \Sigma_Y)$: Y is an $n \times p$ random matrix with mean μ and covariance matrix Σ_Y. By this, we mean that random vector $\delta(Y)$ is normal with mean $\delta(\mu)$ and covariance $\varphi(\Sigma_Y)$, where $\delta(Y)$ and $\delta(\mu)$ represent the coordinate vectors of Y and μ in $\Re^{np}$ with respect to an orthogonal basis for $\mathcal{M}_{n\times p}$ and $\varphi(\Sigma_Y) = E(\delta(Y) - \delta(\mu))(\delta(Y) - \delta(\mu))'$. ($\delta(\cdot)$ and $\varphi(\cdot)$ above may also be written as $[\cdot]$).
$DW_p(m_1, m_2, \Sigma)$: the Wishart–Laplace distribution with (m_1, m_2) degrees of freedom and $p \times p$ nonnegative definite scale matrix Σ, i.e. the distribution of $Z'KZ$, where $Z \sim N_{m\times p}(0, I_m \otimes \Sigma)$ and $K = diag[I_{m_1}, -I_{m_2}]$.

2. Main results

The main theorems of this section require several notions and results some of which arise from the general theory of Jordan algebras. For convenience these notions and results are given in the Appendix (see Lemmas A.1–A.3) and will be referenced when necessary.

We shall now state and prove our main result.

Theorem 2.1. *Suppose that:*

(A1) $Y \sim N_{n\times p}(0, \Sigma_Y)$;
(A2) $\Sigma_{Y'} = L'L$, $L = [L_1, L_2, \ldots, L_p]$, $L_i \in M_{q\times n}$, $q \leq np$;