Journal of Multivariate Analysis 101 (2010) 1284–1295

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Explicit estimators of parameters in the Growth Curve model with linearly structured covariance matrices

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ARTICLE INFO

Article history:
Received 23 March 2009
Available online 7 January 2010

AMS subject classifications:
62H12

Keywords:
Growth Curve model
Linearly structured covariance matrix
Explicit estimators
Residuals

ABSTRACT

Estimation of parameters in the classical Growth Curve model, when the covariance matrix has some specific linear structure, is considered. In our examples maximum likelihood estimators cannot be obtained explicitly and must rely on optimization algorithms. Therefore explicit estimators are obtained as alternatives to the maximum likelihood estimators. From a discussion about residuals, a simple non-iterative estimation procedure is suggested which gives explicit and consistent estimators of both the mean and the linear structured covariance matrix.

1. Introduction

The Growth Curve model introduced by [1] has been extensively studied over many years. It is a generalized multivariate analysis of variance model (GMANOVA) which belongs to the curved exponential family. The mean structure for the Growth Curve model is bilinear in contrary to the ordinary MANOVA model where it is linear. For more details about the Growth Curve model see e.g., [2–5].

In the MANOVA model, when dealing with measurements on k equivalent psychological tests, [6] was one of the first to consider patterned covariance matrices. A covariance matrix with equal diagonal elements and equal off-diagonal elements, i.e., a so-called uniform structure was studied. The model was extended by [7] to a set of blocks where each block had a uniform structure.

Olkin and Press [8] considered a circular stationary model, where variables are thought of as being equally spaced around a circle, and the covariance between two variables depends only on the distance between the variables. Olkin [9] studied a multivariate version of this model in which each element was a matrix, and the blocks were patterned.

More generally, group symmetry covariance models may be of interest since they generalize the above models; see for example [10–12]. In [13] marginal permutation invariant covariance matrices were considered and it was proven that permutation invariance implies a specific structure for the covariance matrices. In particular, shift permutation invariance generates invariant matrices with a Toeplitz structure; e.g., see [14,15].

Furthermore, [16] studied when the covariance matrix can be written as a linear combination of known symmetric matrices but the coefficients of the linear combinations are unknown parameters to be estimated. Chaudhuri et al. [17] considered graphical models and derived an algorithm for estimating covariance matrices under the constraint that certain covariances are zero. As a special case of the model discussed by [17], Ohlson et al. [18] studied banded covariance matrices, i.e., covariance matrices with so-called m-dependence structure.

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0047-259X/$ – see front matter
doi:10.1016/j.jmva.2009.12.023

For the Growth Curve model, when no assumption about the covariance matrix was made, [1] originally derived a class of weighted estimators for the mean parameter matrix. Khatri [19] extended this result and showed that the maximum likelihood estimator also is a weighted estimator. Under a certain covariance structure, [20,21] have shown that the unweighted estimator also is the maximum likelihood estimator. Furthermore, [22] has derived the likelihood ratio test for this type of covariance matrix.

Several other types of structured covariance matrices, utilized by the Growth Curve model, do also exist. For example, Khatri [23] derived the likelihood ratio test for the intraclass covariance structure and [24,25] considered the uniform covariance structure. The autoregressive covariance structure which is natural for time series and repeated measurements have been discussed by [26,27,25].

Closely connected to the intraclass covariance structure is the random effects covariance structure studied by [28,29,21,30,31]. More recently, the random-effect covariance structure have been considered for the mixed MANOVA-GMANOVA models and the Extended Growth Curve models; e.g., see [32–34].

Inference on the mean parameters strongly depends on the estimated covariance matrix. The covariance matrix for the estimator of the mean is always a function of the covariance matrix. Hence, when testing the mean parameters the estimator of the covariance matrix is very important. Originally, many estimators of the covariance matrix were obtained from non-iterative least squares methods. When computer sources became stronger and covariance matrices with structures were considered iterative methods were introduced such as the maximum likelihood method and the restricted maximum likelihood method, among others. Nowadays, when data sets are very large, non-iterative methods have again become of interest.

In this paper we will study patterned covariance matrices which are linearly structured, i.e., see [2], Definition 1.3.7. The goal is not just to obtain reasonable explicit estimators, but also to explore some new inferential ideas which later can be applied to more general models.

The fact that the mean structure is bilinear will result in decompositions of tensor spaces instead of linear spaces as in MANOVA. The estimation procedure which is proposed in this paper will rely on this decomposition. Calculations do not depend on the distribution of the observations, i.e., the normal distribution. However, when studying properties of the estimators the normal distribution is considered.

The organization of this paper is as follows. In Section 2 the main idea is introduced and the decomposition generated by the design matrices is given. In order to support the decomposition presented in Section 2 maximum likelihood estimators for the non-patterned case are presented in Section 3. Furthermore, in Section 4 explicit estimators for patterned covariance matrices in the Growth Curve model are derived. The section will start with a treatment of patterned covariance matrices in the MANOVA model and then it is shown how these estimators can be used when finding overall estimators with the attractive property of being explicit. Finally, some properties of the proposed estimators will be presented in Section 5, and in Section 6 several numerical examples are given.

2. Main idea

Throughout this paper matrices will be denoted by capital letters, vectors by bold lower case letters, and scalars and elements of matrices by ordinary letters.

Some general ideas of how to estimate parameters in the Growth Curve model will be presented in this section. The model is defined as follows.

Definition 2.1. Let $\mathbf{X} : p \times n$ and $\mathbf{B} : q \times k$ be the observation and parameter matrices, respectively, and let $\mathbf{A} : p \times q$ and $\mathbf{C} : k \times n$ be the within and between individual design matrices, respectively. Suppose that $q \leq p$ and $r + p \leq n$, where $r = \text{rank}(\mathbf{C})$. The Growth Curve model is given by

$$\mathbf{X} = \mathbf{ABC} + \mathbf{E}, \tag{1}$$

where the columns of $\mathbf{E}$ are assumed to be independently p-variate normally distributed with mean zero and an unknown positive definite covariance matrix $\boldsymbol{\Sigma}$, i.e., $\mathbf{E} \sim N_{p,n}\left(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{I}_n\right)$.

The estimators of parameters in the model will be derived via a fairly heuristic approach but, among others, the advantage is that it presents a clear way, as illustrated in Section 4, to find explicit estimators of covariance matrices with complicated structures. For estimating parameters in the Growth Curve model we start from the two jointly sufficient statistics, the "mean" $\mathbf{XC}'\left(\mathbf{CC}'\right)^{-}\mathbf{C}$ and the sum of squares matrix

$$\mathbf{S} = \mathbf{X}\left(\mathbf{I} - \mathbf{C}'\left(\mathbf{CC}'\right)^{-}\mathbf{C}\right)\mathbf{X}'. \tag{2}$$

The distribution of the "mean" and the sum of squares matrix are given by

$$\mathbf{XC}'\left(\mathbf{CC}'\right)^{-}\mathbf{C} \sim N_{p,n}\left(\mathbf{ABC}, \boldsymbol{\Sigma}, \mathbf{C}'\left(\mathbf{CC}'\right)^{-}\mathbf{C}\right) \tag{3}$$

and

$$\mathbf{S} = \mathbf{X}\left(\mathbf{I} - \mathbf{C}'\left(\mathbf{CC}'\right)^{-}\mathbf{C}\right)\mathbf{X}' \sim W_p\left(\boldsymbol{\Sigma}, n - r\right), \tag{4}$$