Journal of Multivariate Analysis 102 (2011) 1047–1063

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

A copula-based model of speculative price dynamics in discrete time☆

Umberto Cherubini *, Sabrina Mulinacci, Silvia Romagnoli

University of Bologna, Department of Mathematical Economics, Viale Filopanti 5, 40126 Bologna, Italy

ARTICLE INFO

Article history:
Received 31 May 2009
Available online 5 March 2011

AMS subject classifications:
60J05
60G42
62H20
62P05

Keywords:
Markov processes
Copula function
Efficient Market Hypothesis
Granger causality
H-condition

ABSTRACT

This paper suggests a new technique to construct first order Markov processes using products of copula functions, in the spirit of Darsow et al. (1992) [10]. The approach requires the definition of (i) a sequence of distribution functions of the increments of the process, and (ii) a sequence of copula functions representing dependence between each increment of the process and the corresponding level of the process before the increment. The paper shows how to use the approach to build several kinds of processes (stable, elliptical, Farlie–Gumbel–Morgenstern, Archimedean and martingale processes), and how to extend the analysis to the multivariate setting. The technique turns out to be well suited to provide a discrete time representation of the dynamics of innovations to financial prices under the restrictions imposed by the Efficient Market Hypothesis.

1. Introduction

Finance and physics are the fields in which the theory of stochastic processes has experienced the largest development. In finance, the model of price dynamics first proposed by Bachelier [2] has become well known as the *Efficient Market Hypothesis* (EMH). Simply, a market is called efficient if price changes are not predictable. In its modern version (see [11,26–28]), the model is applied to price logarithm instead of the prices themselves as in the original Bachelier work. In other words, in order to constrain a price S_i, (where i denotes time) to be non-negative, we model the process X_i, which is linked to the price by the relationship $S_i = e^{X_i}$. So, under the EMH, the innovation of the variable X_i cannot be predicted from the current and past values of the variable itself. This is called *weak form efficiency*. If innovation cannot be predicted on the basis of other public information either, the market is said to exhibit *semi-strong efficiency*. If private information is also useless, the market is said to be *strongly efficient* (for a treatment of market efficiency from the point of view of financial econometrics, see chapter 2 in [5]). So, according to the EMH, the price dynamics must be represented by the model $X_i = X_{i-1} + Y_i$, where Y_i represents the innovation, that is the increment of the log-price, at time i. *Weak form efficiency* results in two requirements: (i) X is endowed with the first order Markov property, that is, $\Pr(X_i \le x \mid X_{i-1}, \ldots, X_0) = \Pr(X_i \le x \mid X_{i-1})$ and (ii) the conditional expectation of the increments is equal to zero, that is, $E(Y_i \mid X_{i-1}) = 0$, $\forall i$, which is called the *martingale property*. In more general forms of efficiency, the martingale condition is required to hold true with respect to larger filtrations. This is called *H-condition*.

In the literature, the EMH is enforced by assuming independent increments, namely Lévy and additive processes, which allow a synthetic representation in continuous time. In this paper, we propose a general representation in discrete time that

☆ The authors would like to thank Fabio Gobbi for excellent research assistance with the simulations and three anonymous referees for suggestions that were of great help for the improvement of the paper.

* Corresponding author.
E-mail address: umberto.cherubini@unibo.it (U. Cherubini).

0047-259X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2011.02.004

could exploit the flexibility of copulas. As for the Markov property, a technique to represent first order Markov processes in terms of copulas was first proposed by Darsow et al. [10]. They showed that first order Markov processes are characterized by the following relation

$$C_{X_{j_1},X_{j_2},\ldots,X_{j_n}} = C_{X_{j_1},X_{j_2}} \star C_{X_{j_2},X_{j_3}} \star \cdots \star C_{X_{j_{n-1}},X_{j_n}}$$

where $C_{X_{j_1},X_{j_2},\ldots,X_{j_n}}$ is the copula associated to the vector $(X_{j_1}, X_{j_2}, \ldots, X_{j_n})$ and $C_{X_{j_k},X_{j_{k+1}}}$ stands for the copula function linking X_{j_k} and $X_{j_{k+1}}$ and the $\star$-product operator is defined as

$$A \star B(u, w, v) \equiv \int_0^w \frac{\partial A(u, t)}{\partial t} \frac{\partial B(t, v)}{\partial t}\, \mathrm{d}t \tag{1}$$

for arbitrary bivariate copula functions A and B. This operator allows to express the Chapman–Kolmogorov equation in the language of copulas. Ibragimov [17,18] extended the representation to the case of Markov processes of order k. The same results can obviously be applied to represent a Markov process in k dimensions, and it is in that sense that they will be used in this paper.

Building on the [10] idea much has already been done to explore the temporal dependence features of levels of Markov processes (see [20,6,1,19,3,7]). Already in [20], it has been suggested that a mixture copula could provide a parametric family including "an i.i.d. sequence at one boundary and a perfectly dependent (or persistent) sequence at the other boundary". In spite of this, we are not aware of any contribution addressing the problem of dependence of increments of processes, instead of levels. While on the boundaries of i.i.d. and persistent processes we may have a clear conjecture of what the answer would be, this is not so easy in general cases. This paper proposes a full-fledged method to discriminate Markov processes with independent and dependent increments. As the most simple example, take the copula extracted from Brownian motion, which is one of the cases provided in the original Darsow et al. [10]. This is a typical example of process with independent increments, even though levels have a specific kind of dependence, which can be represented by means of the Gaussian copula

$$C(u, v) = \int_0^u \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(w)}{\sqrt{1-\rho^2}}\right) \mathrm{d}w$$

where $\Phi(x)$ denotes the standard normal distribution and ρ is a correlation parameter. Assuming that the variables in question are two observations of the level of the same process at different times t and s, with $s < t$ this copula naturally denotes a Gaussian dependence structure with $\rho = \sqrt{s/t}$. The question addressed in this paper is how this representation can be generalized to represent all Markov processes with independent increments and how the construction can be amended to extend the class to Markov processes with dependent increments. Apart from this theoretical innovation, we are then interested in investigating whether it is possible to design stochastic processes with dependent increments that abide by the requirements of the EMH, and how the construction can be extended to the multivariate case. From this point of view, this paper contributes to the econometric literature on the use of copula functions for the analysis of financial time series (see [9,24] for a review). This new viewpoint also provides a different extension of financial application beyond that of time-changed Brownian copula, explored in [29,8].

The outline of the paper is as follows. In Section 2, we present the technique used to construct Markov processes with independent and dependent increments. In Section 3, we show how to apply the technique to build several kinds of process. In Section 4, we address the restriction that must be imposed to the dynamics of the process to make it a martingale. Section 5 provides the multivariate extension of the analysis. Section 6 concludes the paper. Proofs that are too lengthy to be included in the text are found in the Appendix.

2. Copula-based Markov processes with (in)dependent increments

We assume a probability space $(\Omega, \Im, \mathbb{P})$ and a sequence of random variables $\{Y_n\}_{n\geq 1}$. We define a discrete time stochastic process $\{X_n\}_{n\geq 0}$ through $X_i = X_{i-1} + Y_i$, $i \geq 1$, assuming, for simplicity, $X_0 = 0$. Moreover, we endow the probability space with a filtration $\{\Im_n\}_{n\geq 0}$ (with $\Im_0$ trivial) to which $\{X_n\}_{n\geq 0}$ is adapted. We denote F_{Y_i}, the cumulative distribution function of the increment Y_i and F_{X_i}, the cumulative distribution function of X_i. Of course, we have $F_{Y_1} = F_{X_1}$. We also assume a set of copula functions C_{X_{i-1},Y_i} representing the dependence structure between the value of the process at the beginning of the period $[t_{i-1}, t_i]$ and its increment during that period. While a first version of the paper provided regularity conditions to ensure that the results would hold true in full generality, here for clarity the paper will focus on absolute continuous copulas; this paper will derive the results in terms of copula density functions when it is simpler to do so. We also assume that all marginal cumulative distribution functions are strictly increasing. Given the copula function linking levels and increments, the task, then, is to determine the temporal dependence structure between X_{i-1} and X_i. The representation of bivariate distributions will allow the full exploitation of the flexibility granted by copula functions

$$\Pr(X_{i-1} \leq x, X_i \leq y) = C_{X_{i-1},X_i}(F_{X_{i-1}}(x), F_{X_i}(y)).$$

Finally, since the stochastic process $\{X_n\}_{n\geq 0}$ is assumed to be a first order Markov process, we may apply the result of [10] to give a complete copula-based description of the law of the process. In order to apply this approach, we start by studying the