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Asymptotic properties of the Bernstein density copula estimator for α-mixing data

Taoufik Bouezmarni [a,b], Jeroen V.K. Rombouts [c,*], Abderrahim Taamouti [d]

[a] Département de mathématiques et de statistique, Université de Montréal, C.P. 6128, succursale Centre-ville, Montréal, Canada, H3C 3J7

[b] Institute of Statistics, Université catholique de Louvain, Canada

[c] Institute of Applied Economics at HEC Montréal, CIRANO, CIRPEE, CORE (Université catholique de Louvain), 3000 Cote Sainte Catherine, Montréal (QC), Canada, H3T 2A7

[d] Departamento de Economía, Universidad Carlos III de Madrid Calle Madrid, 126 28903 Getafe (Madrid), Spain

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ABSTRACT

Copulas are extensively used for dependence modeling. In many cases the data does not reveal how the dependence can be modeled using a particular parametric copula. Nonparametric copulas do not share this problem since they are entirely data based. This paper proposes nonparametric estimation of the density copula for α-mixing data using Bernstein polynomials. We focus only on the dependence structure between stochastic processes, captured by the copula density defined on the unit cube, and not the complete distribution. We study the asymptotic properties of the Bernstein density copula, i.e., we provide the exact asymptotic bias and variance, we establish the uniform strong consistency and the asymptotic normality. An empirical application is considered to illustrate the dependence structure among international stock markets (US and Canada) using the Bernstein density copula estimator.

1. Introduction

The correlation coefficient of Pearson, Kendall's tau, and Spearman's rho are widely used to measure the dependence between variables. Despite their simplicity to implement and interpret, they are not able to capture all forms of dependencies. The copula function has the advantage to model completely the dependence among variables. Further, the above coefficients of dependence can be deduced from the copula. Thanks to [1], the copula function is directly linked to the distribution function. Indeed, the distribution function can be controlled by the marginal distributions, which give the information on each component, and the copula that captures the dependence between components. This fact gives rise to a flexible two step modeling approach where in the first step one models the marginal distributions and in the second step one characterizes the dependence using a copula model.

There are several ways to estimate copulas. First, the parametric approach imposes a specific model for the copula that is known up to some parameters. We can estimate the parameters using the maximum likelihood or inference function for margins. This approach is widely used in practice because of its simplicity, see [2,3] for textbook details. Examples are [4] in the context of survival data, [5] investigates several parametric copulas to model the dependence in time series data, [6] studies the dependence among financial variables for US company data. Parametric copulas are also used in nonparametric regression, see for example [7].

* Corresponding author.
E-mail addresses: jeroen.rombouts@hec.ca (J.V.K. Rombouts), ataamout@eco.uc3m.es (A. Taamouti).

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The second way to estimate copulas is the nonparametric approach. The advantage of this approach is its flexibility to adapt to any kind of dependence structure particular to a given dataset. An important contribution is [8] which suggests the multivariate empirical distribution to estimate the copula function. Empirical copulas are for example used in [9] to test for equality between two copulas. Gijbels and Mielniczuk [10] estimate a bivariate copula using smoothing kernel methods. Also, they suggest the reflection method in order to solve the boundary bias problem of the kernel methods. Chen and Huang [11] propose a bivariate estimator based on the local linear estimator, which is consistent everywhere in the support of the copula function. Rödel [12] uses the orthogonal series method. For independent and identically distributed (*i.i.d.*) data, Sancetta and Satchell [13] develop a Bernstein polynomial estimator of the copula function and find an upper bound of its asymptotic bias and variance. They also establish the asymptotic normality of the Bernstein density copula estimator, however without an explicit expression of the variance.

In this paper, we consider α-mixing dependent data and propose nonparametric estimation of the density copula based on Bernstein polynomials. We suppose that the marginal distributions are estimated by their empirical distribution functions. We focus only on the dependence structure between the stochastic processes, captured by the copula density defined on the unit cube, and not the complete distribution. We study the asymptotic properties of the Bernstein density copula, i.e., we provide the exact asymptotic bias and variance, and we establish the uniform strong consistency and the asymptotic normality of Bernstein estimator for the density copula. In an empirical application, we consider two international stock market indices to illustrate the flexibility of the Bernstein copula by comparing it to parametric copulas.

Motivated by Weierstrass theorem, Bernstein polynomials are considered by Lorentz [14] who proves that any continuous function can be approximated by Bernstein polynomials. For density functions, estimation using the Bernstein polynomial is suggested by Vitale [15] and with a slight modification by Grawronski and Stadtmüller [16]. Tenbusch [17] investigates the Bernstein estimator for bivariate density functions and Bouezmarni and Rolin [18] prove the consistency of Bernstein estimator for unbounded density functions. Kakizawa [19,20] consider the Bernstein polynomial to estimate density and spectral density functions, respectively. Tenbusch [21] and Brown and Chen [22] propose estimators of the regression functions based on the Bernstein polynomial. In the Bayesian context, Bernstein polynomials are explored by Petrone [23,24], Petrone and Wasserman [25], and Ghosal [26].

This paper is organized as follows. The Bernstein copula estimator is introduced in Section 2. Section 3 provides the asymptotic properties of the Bernstein density copula estimator, that is the asymptotic bias and variance, the uniform strong consistency, and the asymptotic normality for α-mixing dependent data. Section 4 presents an empirical illustration using financial data and Section 5 concludes the paper.

2. Bernstein copula estimator

Let $X = \{(X_{i1}, \ldots, X_{id})', i = 1, \ldots, n\}$ be a sample of n observations of α-mixing vectors in $\mathbb{R}^d$, with distribution function F and density function f. A sequence is α-mixing of order h if the mixing coefficient $\alpha(h)$ goes to zero as the order h goes to infinity, where

$$\alpha(h) = \sup_{A\in\mathcal{F}_1^t(X), B\in\mathcal{F}_{t+h}^\infty(X)} |P(A\cap B) - P(A)P(B)|,$$

$\mathcal{F}_1^t(X)$ and $\mathcal{F}_{t+h}^\infty(X)$ are the σ-field of events generated by $\{X_l, l \le t\}$ and $\{X_l, l \ge t+h\}$, respectively. The concept of α-mixing is omnipresent in time series analysis and is less restrictive than β- and ρ-mixing. The following condition requires an α-mixing coefficient with exponential decay. We assume that the process X is α-mixing such that

$$\alpha(h) \le \rho^h, \quad h \ge 1, \tag{1}$$

for some constant $0 < \rho < 1$.

According to Sklar [1], the distribution function of X can be expressed via a copula:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \tag{2}$$

where F_j is the marginal distribution function of the random variable $X^j = \{X_{1j}, \ldots, X_{nj}\}$ and C is a copula function which captures the dependence in X. Deheuvels [8] uses a nonparametric approach, based on the empirical distribution function, to estimate the distribution copula. Using Bernstein polynomials, to smooth the empirical distribution, Sancetta and Satchell [13] propose the empirical Bernstein copula which is defined as follows: for $s = (s_1, \ldots, s_d) \in [0, 1]^d$

$$\hat{C}(s_1, \ldots, s_d) = \sum_{\upsilon_1=0}^{k} \cdots \sum_{\upsilon_d=0}^{k} F_n\left(\frac{\upsilon_1}{k}, \ldots, \frac{\upsilon_d}{k}\right) \prod_{j=1}^{d} P_{\upsilon_j,k}(s_j), \tag{3}$$

where k is an integer playing the role of the bandwidth parameter, F_n is the empirical distribution function of X, and $P_{\upsilon_j,k}(s_j)$ is the binomial distribution function:

$$P_{\upsilon_j,k}(s_j) = \binom{k}{\upsilon_j} s_j^{\upsilon_j} (1-s_j)^{k-\upsilon_j}.$$